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BULLETIN OF THE

NEW YORK MATHEMATICAL SOCIETY

EVOLUTION OF CRITERIA OF CONVERGENCE.

BY PROF. FLORIAN CAJORI.

THE expressions convergent and divergent series were used for the first time in 1668 by James Gregory. Newton and Leibniz felt the necessity of inquiring into the convergence of infinite series, but they had no proper criteria, excepting a test advanced by Leibniz for alternating series. By Euler and his contemporaries the *formal* treatment of series was greatly extended, while the necessity for determining the convergence was generally lost sight of. To be sure, it was Euler who first observed the semi-convergence of a series. He, moreover, remarked that great care should be exercised in the summation of divergent series. But this warning was not taken so seriously by him as it would be by a modern writer, for in the very same article* in which it occurs Euler did not hesitate to write

$$\ldots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \ldots = 0,$$

simply because

$$n + n^2 + \ldots. = \frac{n}{1-n}; \; 1 + \frac{1}{n} + \frac{1}{n^2} + \ldots = \frac{n}{n-1}.$$

The facts are that Euler reached some very pretty results in infinite series, now well known, and also some very absurd results, now quite forgotten. Protests were made by Nicolaus Bernoulli and Varignon against the prevailing reckless use of series: isolated attempts at establishing criteria of convergence are on record: but the dominating sentiment of the age frowned down any proposition which would put limitations upon operations with series. The faults of this period found their culmination in the Combinatorial School in Germany, which has now passed into deserved oblivion.

I. *Special Criteria.* In the progress of mathematics, truth

* *Comm. Petrop.*, vol. 11, p. 116.

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criteria, because they all depend upon a comparison of the nth term of the series with special functions, a^n, n^κ, $n(\lg n)^\kappa$, etc. Kummer's article * antedates the papers of De Morgan and Bertrand. He establishes the theorem that the series Σu_p is convergent if a function $\varphi(p)$ can be found such that $\lim \varphi(p) \cdot u_p = 0$ and $\lim \left\{ \varphi(p) \frac{u_p}{u_{p+1}} - \varphi(p+1) \right\}$ is not equal to zero. Raabe's test can be deduced from this. Du Bois Reymond divides all criteria into two classes : criteria of the *first kind* and criteria of the *second kind*, according as the general term u_p or the ratio of u_{p+1} and u_p is made the basis of research. Kummer's looks like a mixed criterion, but it is really of the second kind. The true significance of the second part of that criterion was at first overlooked. Bertrand says in his *Calcul différentiel* (vol. 1, p. 244) that the great indeterminateness in the mode of applying it is an advantage to those who know how to profit thereby. But Du Bois Reymond † points out that there is nothing indeterminate, that in the selection of the function φ we have practically the choice only between the quantities $L_\kappa(n)$ arising out of the criteria of the second kind, and if criteria of the second kind were discovered, affording more delicate tests than do the logarithmic ones, such new criteria would be embraced by the second part of Kummer's criterion. $\text{Lim} \left\{ \varphi(p) \frac{u_p}{u_{p+1}} - \varphi(p+1) \right\}$ is therefore a necessary general form of all criteria of the second kind which renders an investigation of the first part of Kummer's criterion, viz. $\lim u_p \cdot \varphi(p)$, superfluous. The above criterion, bereft of the first part, was invented anew, over half a century after its first publication, by Jensen‡, who was unaware of Kummer's researches.

Dini generalized Kummer's result, but his paper § (known to the writer only through the remarks upon it made by Pringsheim) remained unnoticed, and the same ground was traversed independently six years later by Du Bois Reymond, though with somewhat greater thoroughness. The investigations of both rest upon a comparison of the terms u_n of a series with the expression $\psi(n) - \psi(n+1)$, where $\psi(n)$ invariably either increases or decreases with n. Du Bois Reymond makes the following general statement, from which the useful criteria of the first kind can be deduced as special cases : If the terms u_n of any series of positive terms be brought to the form

$$\frac{1}{\lambda n} \left\{ \psi(n) - \psi(n+1) \right\},$$

* *Crelle*, vol. 13 (1835).
† *Crelle*, vol. 76 (1873).
‡ *Comptes Rendus*, vol. 106 (1888).
§ *Annali dell' Univ. Tosc.*, vol. 9.

BULLETIN

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expressed in several equivalent forms of which the following are two.

$$\Sigma a_\nu \begin{cases} \text{diverges if } a_{\nu+p} \geq h \,.\, d_\nu, \text{ or if } \lim \dfrac{a_{n+p}}{d_n} \geq g > h \\ \text{converges if } a_{\nu+p} \leq H \,.\, c_\nu, \text{ or if } \lim \dfrac{a_{n+p}}{c_n} \leq G < H \end{cases}$$

where $n = \infty$, h and H are finite positive quantities, $d_\nu = D_\nu^{-1}$ is the general term of a divergent series, $c_\nu = C_\nu^{-1}$ is the general term of a convergent series and p represents any constant positive integer. After proving that the general term of any divergent series Σa^ν, can be expressed in the form $\dfrac{M_{\nu+1} - M_\nu}{M_\nu}$, and that the term of any convergent series Σa_ν can be expressed in the form $\dfrac{M_{\nu+1} - M_\nu}{M_{\nu+1} \,.\, M^\nu}$, and where M_ν is positive and finite for finite values of ν, and invariably increases with ν, being ∞ when ν is ∞, he writes the above general criterion thus,

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim \dfrac{M_n}{M_{n+1} - M_n} \,.\, a_{n+p} \geq g > 0 \\ \text{converges if } \lim \dfrac{M_{n+1}\, M_n}{M_{n+1} - M_n} \,.\, a_{n+p} \leq G < \infty \end{cases}$$

Putting $M_\nu = \nu$, and replacing ν successively by $\lg \nu$, $\lg_2 \nu, \ldots \lg^\kappa \nu$, he deduces Bertrand's and Bonnet's forms of logarithmic criteria and then reaches a remarkable generalization constituting a new general criterion of the first kind, analogous to Kummer's criterion of the second kind, viz., the series Σa_ν is always convergent if a positive number $\varphi(\nu)$ exists for which

$$\lim \frac{\lg \dfrac{1}{\varphi(n)\, a_{n+p}}}{\sum\limits_0^n{}_\nu \dfrac{1}{\varphi(\nu)}} > 0$$

The other special criteria are also deduced from the general form. The general type of criteria of the second kind is

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim P_n \left(D_n a_{n+p} - D_{n+1} a_{n+p+1}\right) < 0 \\ \text{converges if } \lim P_n \left(C_n a_{n+p} - C_{n+1} a_{n+p+1}\right) > 0 \end{cases}$$

where P_n may be any positive factor. If we take

$P_n = \frac{1}{a_{n+p+1}}$, the criteria become

$$\Sigma a_\nu \begin{cases} \text{diverges if } \lim \left(D_n \frac{a_{n+p}}{a_{n+p+1}} - D_{n+1} \right) < 0 \\ \text{converges if } \lim \left(C_n \frac{a_{n+p}}{a_{n+p+1}} - C_{n+1} \right) > 0 \end{cases}$$

From this Kummer's and the special criteria of the second kind are deduced, including those of Gauss.

As regards the scope of criteria of the first and of the second kind, we find that the first will always be decisive whenever the second are, but the second may fail ever so often when the first do not. Though more limited, the criteria of the second kind are nevertheless of value, for they often yield results more easily and quickly. This narrower range of application is due to the fact first clearly pointed out by Pringsheim that the terms a_ν of a series may lie always above or always below the corresponding terms b_ν of another series and yet there may be no fixed relation whatever between the ratios $\frac{a_{\nu+1}}{a_\nu}$ and $\frac{b_{\nu+1}}{b_\nu}$. Since in a series of positive terms the order in which the terms come has nothing to do with the convergence or divergence of the series, it is clear that $\frac{a_{n+1}}{a_n}$ does not usually approach a limit. Thus the case $\frac{a_{\nu+1}}{a_\nu} >$ or $< \frac{b_{\nu+1}}{b_\nu}$ is only a very special one, and the probability that the series Σa_ν, yielding to the tests of the first kind, yield also to those of the second kind, is very small.

The range of special tests of the first kind has been partly considered in connection with logarithmic criteria. Pringsheim points out that the logarithmic criteria have been much overvalued, that they are applicable only to series whose terms are *essentially* decreasing, so that the increase or decrease in the values of terms does not exceed certain limits. The terms must be always above or always below the corresponding terms in $\Sigma \frac{g}{L_\kappa(\nu)}$ or $\Sigma \frac{G}{L_\kappa(\nu) \lg_\kappa^\rho(\nu)}$. But this property depends again upon the order of the terms. Suppose a certain order is favorable for the use of these tests, then a promiscuous displacement of terms may render the logarithmic scale or any other scale wholly inapplicable, and all this without altering the convergence or divergence, or even the sum of the series. This failure is in no way due to the *form* of the

general criteria, for Pringsheim shows that for every Σa_ν, *practical* criteria of the first and second kind do exist; but his proof of this fact yields no method of finding them, except when he knows beforehand the very thing to be determined, namely, whether Σa_ν be convergent or not!

In addition to the criteria of the first kind and second kind, Pringsheim establishes an entirely new criterion of a *third kind* and also *generalized criteria of the second kind*, which apply, however, only to series with never increasing terms. Those of the third kind rest mainly upon the consideration of the limit of the difference, either of consecutive terms or of their reciprocals. In the generalized criteria of the second kind he does not consider the ratio $\frac{a_{\nu+1}}{a_\nu}$ of two consecutive terms, but the ratio of any two terms, however far apart, and deduces, among others, two criteria previously given by Kohn * and Ermakoff,† respectively.

COLORADO COLLEGE, *April* 12, 1892.

NOTE ON AN ERROR IN BALL'S HISTORY OF MATHEMATICS.

BY DR. ARTEMAS MARTIN.

I DESIRE to call attention to what seems to me to be an error in Ball's "Short History of Mathematics," page 102, concluding clause of last paragraph, where the author ascribes to Diophantus the statement "that the sum of three square integers can never be expressed as the sum of two squares."

That the above statement is not in accordance with the facts is evident, since

$$(q^2+r^2-s^2-u^2)^2+(2qu)^2+(2ru)^2=(q^2+r^2-s^2+u^2)^2+(2su)^2$$

identically, no matter what values be assigned to q, r, s, u.

If we take $q = 1$, $r = 2$, $s = 3$, $u = 4$, then, after dividing all the numbers by 4^2, we have

$$2^2 + 4^2 + 5^2 = 3^2 + 6^2 = 45.$$

Let $q = 1$, $r = 2$, $s = 4$, $u = 3$, and we find, after dividing by 2^2,

$$3^2 + 6^2 + 10^2 = 1^2 + 12^2 = 145, = 8^2 + 9^2.$$

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* *Grunert's Archiv*, vol. 67, pp. 63–95.

† *Darboux's Bulletin*, vol. 2, p. 250; vol. 17, p. 142.

By varying the values of q, r, s, u, an infinite number of sets of three square integers can be found that can also be expressed as the sum of two squares; but the first given above is probably the *smallest* set in which the numbers are all different.

Since writing the foregoing note I have been informed by Mr. Ball that the above error probably arose from a confused memorandum of Diophantus's impossible forms for the sums of two and three squares.

WASHINGTON, D. C., *September* 15, 1892.

THE THEORY OF EQUATIONS.

An Elementary Course in the Theory of Equations. By C. H. CHAPMAN, Ph.D., Associate in Mathematics in Johns Hopkins University. New York, John Wiley & Sons, 1892. 12mo, pp. viii + 90.

IN the past no American text-book has treated exclusively of the theory of equations, but the subject has been presented briefly in the closing chapters of the more complete works on algebra. The marked tendency at the present time to prefer a sort of parallel study of several correlated branches of mathematics to carrying on successively each subject as far as possible before taking up another, has led to the introduction in many cases of short courses in trigonometry, analytical geometry, and calculus between the more elementary algebra and the theory of equations. Under this system it becomes at once desirable to treat the latter subject in a separate text-book, into which it is proper to introduce the notation and principles of the infinitesimal calculus, although logically they may be altogether unnecessary.

The first impression conveyed by the little book which comes from the pen of Dr. Chapman is that it has been unduly condensed, but upon a more careful examination one perceives that to a student possessing a very slight knowledge of analytical geometry and calculus the treatment of the subject is unusually intelligible. The book is worthy of praise both on scientific and on pedagogical grounds. The definitions are accurate, the demonstrations rigorous, and the choice of material excellent; while the questions addressed to the student at various points, the problems, and suggested demonstrations are sure to be effective.

The first part relates to determinants. It occupies but twenty one pages, and leads from first principles through the most elementary propositions to the solution of linear equa-

tions, and elimination by Sylvester's dialytic method. In this connection we could wish that the author should have suggested to the student the application of elimination to the rationalization of algebraic equations, thus showing that the equation treated in the second part actually represents the most general algebraic equation possible. The second part contains forty pages. After explaining the graphical representation of imaginaries, it gives the demonstration that every equation has a root by the method of Gauss. Then follow symmetric functions, the proof that they can be expressed by means of the coefficients, and their application to the formation of resultants and discriminants. Here Sylvester's method reappears for the purpose of shedding a side-light on the last mentioned subject. Twenty four pages suffice for the third part. They contain Descartes' rule of signs, Rolle's theorem, the method of freeing an equation from equal roots, Sturm's theorem, and Horner's method. One section gives a rule for finding the superior limit of the real roots, but, strange to say, neglects to mention the obvious modification for finding their inferior limit.

It will be seen that the contents of the work are amply sufficient for the undergraduate curriculum of any college or scientific school, although they are made to occupy only ninety pages. At the end of the book is found a note on rectangular arrays, but it is not easy to see how this note, appearing alone, can have any significance to the students in whose hands the book may be placed.

THOMAS S. FISKE.

NOTES.

THE officers of Section A at the Rochester Meeting of the American Association for the Advancement of Science were: Vice-President, J. R. Eastman of Washington; Secretary, Winslow Upton of Providence. The following papers were read: The neglected field of fundamental astronomy, by J. R. Eastman; On the conflict of observation with theory as to the earth's rotation, by S. C. Chandler; Meteorological observations made in April, 1890, 1891, 1892 in the totality path of the eclipse of April 16, 1893, by David P. Todd; Latitude of the Sayre Observatory, by C. L. Doolittle; The secular motion of a free magnetic needle, by L. A. Bauer; On the discriminators of the discriminant of an algebraic equation, by Mansfield Merriman; The spectroheliograph of the Kenwood Astro-Physical Observatory, Chicago, and results obtained in the study of the sun, by G. E. Hale; Least square fallacies, by T.

H. Safford; Models and machines for showing curves of the third degree, by A. W. Phillips; On the imaginary of algebra, by Alexander Macfarlane; Differential formulæ for orbit corrections, by T. H. Safford; The iced-bar base apparatus of the U. S. Coast and Geodetic Survey, by R. S. Woodward; On the construction of a prime vertical transit instrument for the determination of the latitude of Harvard College Observatory, by W. A. Rogers; Lineo-linear vector functions, by A. S. Hathaway; Thermal absorption in the solar atmosphere, by E. B. Frost; Electric lights for astronomical instruments, by J. E. Kershner; Forms of solar faculæ, by G. E. Hale; European observatories, by J. A. Brashear; List of 30 new proper motion stars, by C. L. Doolittle; Proper motions of 89 stars within 10° of the North Pole with remarks on the present state of the problem of solar motion, by T. H. Safford; Concerning a congruence-group of order 360 contained in the group of linear fractional substitutions, by E. H. Moore; On the intersection of an equilateral hyperbola and the sides of a plane triangle (a question in trilinears), by William Hoover; Practical rules for testing whether a number is divisible by 7, or any other small prime; and if not divisible, to ascertain the remainder; also Increase in constant for addition in testing for integral values in the equation of quarter squares, by James D. Warner; On the general problem of least squares, by R. S. Woodward. The next general meeting will be held in Madison, Wis. The officers are: President, William Harkness of Washington; Vice-President of Section A, C. L. Doolittle of South Bethlehem, Pa.; Secretary of Section A, A. W. Phillips of New Haven.

A SCIENTIFIC CONGRESS is being arranged in connection with the World's Fair at Chicago in 1893. The general committee on mathematics and astronomy has issued a preliminary address inviting the cooperation of all persons and societies interested. The members of this committee are: George W. Hough (Chairman, and in special charge of *astronomy*), Elias Colbert (Vice-Chairman), E. H. Moore (in special charge of *pure mathematics*), G. E. Hale (in special charge of *astrophysics*), G. A. Douglas, Malcolm McNeill, R. W. Pike, G. C. Comstock, and W. W. Payne. Partial lists of the advisory councils contain the names of several members of the NEW YORK MATHEMATICAL SOCIETY, viz.: Arthur Cayley, Thomas Craig, H. A. Howe, Simon Newcomb, J. K. Rees, T. H. Safford, G. M. Searle, Ormond Stone, R. S. Woodward and C. A. Young.

THE Columbia College Library has purchased from Otto Struve, former director of the Pulkowa Observatory, his

entire collection of astronomical and physical works, containing 4,361 books and 3,056 pamphlets. The astronomical department hopes to publish in the near future a catalogue of its library for the use of investigators.

DR. ARTEMAS MARTIN submits the following correction of an error in the Royal Society Catalogue of Scientific Papers, vol. IX., 1874–1883:

"Of the papers accredited on p. 790 to Ezekiel Brown Elliott only Nos. 4, 12 and 13 are by him; the others, Nos. 5–11, 14–17, are by Edwin Bailey Elliott, Queens College, Oxford, England, as may be seen by consulting the references.

"Ezekiel Brown Elliott died May 24, 1888. He was in the employ of the U. S. Government in various capacities for many years, and at the time of his death occupied the position of Government Actuary in the U. S. Treasury, to which position he was appointed in July, 1881. It seems remarkable that so well-known a mathematician as Edwin Bailey Elliott should be confounded with Ezekiel Brown Elliott, and all of the former's papers credited to the latter by the compiler of the Royal Society Catalogue, Edwin Bailey Elliott not appearing in the Catalogue at all."

ARRANGEMENTS are being made at Göttingen for the purpose of carrying out a plan which was adopted shortly after the death of Wilhelm Weber, and which has in view the erection of a joint memorial to Gauss and Weber in recognition of their contributions to physical science.

THE meeting of the *Deutsche Mathematiker-Vereinigung* which was to have been held this year at Nuremberg, has been given up on account of the cholera epidemic in Germany.

F. TISSERAND has been appointed director of the Observatory of Paris to succeed the late Admiral Mouchez.

THE French Academy of Sciences has elected Professor von Helmholtz foreign member to succeed the late Emperor Dom Pedro of Brazil, and Professor Sophus Lie corresponding member in the place of the late Professor Kronecker.

ON May 29 last Professor Carl Heinrich Schellbach died at Berlin in his eighty-eighth year; and on June 22 at Paris, Professor Pierre Ossian Bonnet, member of the French Institute, in his sixty-third year.

PROFESSOR E. H. MOORE of Northwestern University has been called to the University of Chicago as Professor of Mathe-

matics. He will have associated with him as Associate Professor, Dr. Oskar Bolza, recently of Clark University, who is to begin active work on January 1st, and as Assistant Professor Dr. Heinrich Maschke, recently of the Weston Electrical Company. Mr. J. W. A. Young will be a tutor in the same department.

DR. H. S. WHITE, of Clark University, has been appointed Associate Professor of Mathematics at Northwestern University, Evanston, Ill.

DR. L. S. HULBURT has been appointed Instructor in Mathematics at Johns Hopkins University.

T. S. F.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

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CAMPBELL (W. W.). Practical Astronomy. Ann Arbor 1892. 8vo. With illustrations. $1.50

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. (In 3 Bänden.) Band II: Von 1200 bis 1668. Theil 2. Leipzig 1892. gr. 8. pg. 10 u. 501—863 m. Holzschnitten. M. 10.00

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DUPUIS (N. F.). The principles of Elementary Algebra. New York, Macmillan, 1892. 12mo, pp. 344. $1.10

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all numbers from 1 to 125000, and the values of S and T used to obtain the sines and tangents of small arcs. This is followed by the trigonometric table, in which the functions are given for every microgone. Eight decimals are used for each microgone up to 25000, after which the eighth decimal is omitted from all but the centimiligones.

In order to secure accuracy the author has made a comparison with the famous manuscript tables of Prony, preserved at Paris. The proof sheets were afterwards compared with the tables already mentioned as having been issued recently by the French government. Moreover, a final check was secured by comparing in every case the sum of the cosine and tangent with the sine. In the course of his comparisons M. Mendizábal-Tamborrel found eight errors in the French tables. These have been communicated by him to the *Académie des Sciences* at Paris. Much praise is due the author for the excellent manner in which he has dealt with the difficult problem of arranging the logarithms upon the pages. Old style type is used, and suitable tables of proportional parts are given whenever possible.

In a short notice like the present, it does not seem desirable to enter upon the question of the best method of angular measurement. But it is very important to have extended tables in existence arranged according to the various proposed methods: and to the author of the present work is therefore due the credit of having performed a considerable service to science. That this opinion is very generally held, is sufficiently shown by the list of subscribers to the present work, which includes the names of nearly all the most distinguished astronomers and geodesists. HAROLD JACOBY.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, October 8, at half-past three o'clock, the president in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Professor Levi Leonard Conant, Worcester Polytechnic Institute; Dr. George Daniel Gable, Lafayette College; Professor William Jasper Kerr, State University of Utah; Mr. John Cunningham McLennan, University of Toronto; Mr. Joseph Clement Pfister, Columbia College. The secretary announced the publication and distribution since the last meeting of a pamphlet containing a new list of members, the constitution, by-laws, etc.

The following papers were read: "An orthomorphic transformation of the ellipsoid," by Miss Ella C. Williams; "Some geometrical illustrations of theorems in number," by Professor Ellery W. Davis. In the former paper the elliptic functions in Weierstrass's form were employed to obtain formulas for the transformation of the surface of an ellipsoid into a plane rectangle in such a manner that corresponding infinitesimal elements should be similar to each other. The latter paper has been contributed by the author to the *American Journal of Mathematics.* T. S. F.

Dr. Robert Fricke and Dr. Friederich Pockels, both of whom have recently published books under Professor Klein's guidance, have become *privatdocenten* at the University of Göttingen. M. BÔ.

R. de Paolis, professor of higher geometry at the University of Pisa, died recently at Rome, at the early age of thirty-eight years.

Dr. A. E. Hess, of the University of Marburg, has become *ordinarius* professor of mathematics there, and Dr. H. Minkowski has received a professorship of mathematics at Bonn.

The French Association for the Advancement of Science commenced its twenty-first annual meeting at Pau on September 17. H. J.

The department of mathematics at the University of Chicago announces for the present autumn quarter, in the graduate department, a course on the theory of the icosaedron by Professor Moore; in the higher two classes of the undergraduate department, a course in advanced calculus and one in the theory of equations and invariants, both by Professor Maschke; also one in the theory of substitutions, with applications to algebraic equations, by Mr. Young. All these courses occupy four hours per week each.

A preliminary announcement for the winter quarter gives in the graduate department to Professor Moore a seminary occupying four hours a week and devoted to (a) geometry of space of n dimensions, (b) configurations, (c) groups of operations (the abstract general theory; particular group-forms), and (d) linear automorphic functions, also a course on the elliptic modular functions; to Professor Bolza a course on linear differential equations and one on the theory of functions (introductory); to Mr. Young a course on the theory of numbers. In the higher classes of the undergraduate department Professor Maschke is to follow the advanced

calculus with differential equations and is to continue the theory of invariants; while Mr. Hancock is to begin an elementary course in the calculus.

In the spring quarter Dr. Bolza expects to give elliptic functions in the place of linear differential equations. The other graduate courses, with the exception of Mr. Young's, will continue as in the preceding term. In the undergraduate department Professor Maschke will lecture on higher plane curves and on analytical mechanics, and Mr. Hancock will continue his course in calculus. The instruction in the lower two classes of the college will be given, it is announced, by Mr. Young and Mr. Hancock. T. S. F.

The graduate courses in mathematics given the present year at Princeton are: by Professor Fine, (1) differential equations, general course, embracing the elements of the theory of linear equations and of Lie's transformation theory, (2) theory of substitutions and the arithmetical theory of the algebraic equation; by Professor Thompson, (3) theory of functions, (4) elliptic functions and their applications to geometry and dynamics. H. B. F.

In the various courses in astronomy at the Johns Hopkins University there are now eighteen advanced students, five of whom intend making astronomy their principal subject for the doctor's degree. Professor Newcomb is giving a course in celestial mechanics to the most advanced students. Dr. Poor is giving two courses; the first includes a careful discussion of precession and nutation, to be followed by the methods of computing orbits and special perturbations: the second is a less advanced course and includes the principal problems of spherical and practical astronomy, a general outline of the problems of theoretical, and a sketch of historical astronomy. C. L. P.

Professor H. A. Newton of Yale has been elected a foreign member of the Royal Society of London. Among other Americans recently elected to such membership are Professor H. A. Rowland and Dr. B. A. Gould. T. S. F.

No. 12.

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MATHEMATICAL RECREATIONS.

Mathematical Recreations and Problems of Past and Present Times. By W. W. ROUSE BALL, Fellow and Lecturer of Trinity College, Cambridge; and of the Inner Temple, Barrister-at-Law. London: Macmillan and Co., 1892. 8vo, pp. xii. + 240.

PERHAPS "mathematical recreations" can be more or less sharply classified under two heads:

1. Particular problems treated by general mathematical theories and methods, but so that the layman may not recognize the general in the particular; as with those puzzles which really turn upon the solution of simple equations.

2. Theories or parts of theories, small and distinct enough for popular presentation, yet mathematical in the sense that, as with the Chinese rings or the tower of Hanoï, they deal with number, form and relation by consecutive reasoning whose methods, as far as independent of subject-matter, are essentially those of mathematics; perhaps mathematical too in the sense that, as with some topological studies useful toward function-theory, they are already auxiliary to larger doctrines recognized as main parts of the science, or that, as with some apparently disconnected properties of numbers, they bid fair to be included later in general theories.

Such "recreations," whether relating to pure or to applied mathematics, would naturally, as compared with most other mathematical studies, be more concrete, and would appeal rather less formally to the logical sense, and relatively more to imagination and ingenuity. But we must not forget Kirkman's caution, uttered in connection with his school-girl problem: "A solution . . . . , to be mathematical and not tentative, must inform us whether or no, and why, [so and so] will . . . . be suitable; and [this] before trial and examination of the [particular solution]"—or at least, *independently* of such trial. It is commonly all-important "to see the secret of the symmetry." *

These studies may be of more value than we realize: not only as pleasant introductions to existing theories, but yet more as leading the amateur by short paths to new fields where, not merely as a gleaner, he may experience the satisfaction of adding to our present knowledge something which is of interest in itself, is largely mathematical in spirit and method, and may some day conduce to the great generalizations of mathematics proper. As the world's leisure increases, a certain democracy in science may be essential to the right

* *Cambridge and Dublin Math. Journal*, vol. 5, 260.

mental and moral development of the race; a democracy which the rapidly increasing sweep, abstractness and subtilty of the older sciences might hinder were not fresh shoots continually thrown out nearer to the ground in the form of "recreations."

In the attractive little book before us, the phrase "Mathematical Recreations" appears to be used, consistently, nearly as above; all the problems having a mathematical bearing. The phrase formerly included scraps of almost everything; much as, etymologically, "Mathematics" is synonymous with "Science." Part of the title pages of Leake's edition of Van Etten (London, 1674) runs thus: "*Mathematical Recreations: or, a Collection of Sundrie excellent Problems, out of the Ancient and Modern Philosophers; as, Secrets and Experiments in Arithmetick, Geometry, Cosmography, Horologiography, . . . Musick, ... Chymistry, Water-works, Fire-works, &c., not vulgarly manifest till now;*"—and among the "mathematical" secrets here revealed are "How to charge a cannon without Powder," "How to make an infallible perpetual Motion," "How to practise excellently the reanimation of Simples," "How to make a Cone or Pyramidal Body move upon a Table" [by putting a live beetle under it!], "How out of a quantity of Wine to extract that which is most windy and evil, that it hurt not a sick person", and other recipes for domestic and magic art.

The present work is, in part, a brief summary by sample of what is mathematical and best in the three traditions of Bachet (1612, 1624), Leurechon-Mydorge (1630) and Van Etten-Oughtred (1653): a tradition continued by Ozanam, Montucla, and Hutton (1803, 1840). But this is supplemented by abundant material taken from the classical writers, from Fermat down, and from articles in scientific journals, especially of the past fifteen years. Good use is also made of the author's private note-book, and of Lucas's elaborate *Récréations Mathématiques* (1882–3). The work is a very judicious and suggestive compilation, not meant mainly for mathematicians, yet made doubly valuable to them by copious references. The style, in the main, is so compact and clear that what is central in a long argument or process is admirably presented in a few words: yet there is occasional ambiguity or inaccuracy, due now and then perhaps to fundamental misconceptions, but oftener to a kind of carelessness which one feels all the more constrained to notice, even at the risk of appearing hypercritical, just because it is not uncommon with otherwise good writers.

One great merit of this or any other really good book on such a subject is its suggestiveness: in running through its pages, one is pretty sure to think of additional problems on

the same general lines. I will indicate a few such as we go on, in the hope that some reader may be tempted to work them up.

In the first two chapters are various elementary tricks, problems, paradoxes, games and puzzles; of arithmetic, geometry and topology. The specimen methods of finding a number thought of, are of course very simple : mere solution of linear equations, or application of elementary theory as to integers and factors or as to the Arabic notation ; yet even here an instructive problem suggests itself to the reader, viz. : How to classify all such methods. So with the next trick, to find the result of a series of operations performed on an unknown number, *i.e.*, so to arrange the operations that the number disappears from the result ; and so when two or more numbers unknown to the conjurer have been thought of, or when an unknown selection or arrangement has been made from numbers known to him. Hints are given as to how these tricks may be made more mysterious, and may be connected with numbered things like dials, cards, etc. Then come, among other things, the counting-out problem ; the weights problem with Macmahon's generalizations; and a specimen of what we may describe as the puzzle of dividing $a + b + \ldots$ ounces of balsam into portions of a, b, ... ounces, when one has only k jars, holding m_1, m_2, ... m_k ounces respectively. Chapter I. closes with an interesting account of some of Fermat's and Mersenne's speculations, especially as to the prime factors of $2^m \pm 1$. Among other good reasons given for suspecting that Fermat had some method, now lost, for finding the factors of large numbers, is this : that, in reply to a question from Mersenne as to whether 100895598169 had factors, Fermat wrote, apparently by the next day,* that they were the primes 112303 and 898423. And yet, from the prominence of numbers $2^m \pm 1$ in his own and Mersenne's researches, might not Fermat naturally suspect that the factors sought had some such form ; or at least, what is true, that one factor was nearly 2^m times the other ? If so, he would probably think to divide the given number by 2, 4, 8 . . . . , and to compare the quotient with the nearest perfect square : a procedure certain to have given the actual factors at the first trial.

Chapter II. begins with good samples of two kinds of geometric fallacy : what we may call the *topological*, where one hastily assumes (*e.g.*) that a certain intersection lies above and not below a certain line ; and the cruder *optical*, where in the

* Fermat wrote: "Vous me démandez . . . une méthode pour découvrir, dans l'espace d'un jour, s'il est premier ou composé," etc. (*Lucas*, vol. 2, p. 230).

absence of strict geometric analysis a slight inaccuracy of form goes undetected and the eye deceives us into thinking (*e.g.*) that a square 8×8 inches has been dissected into a rectangle 5×13. It might be of interest if some one would classify, exhaustively under his own chosen restrictions, all the great subdivisions of these and kindred geometric fallacies; and the like may be said of algebraic fallacies. Such classification would be the more useful because a resolved paradox is always instructive and may lead, by inversion or the *reductio ad absurdum*, to positive results, though perhaps only to very simple ones.

Then come the determination of the number of colors needed to keep all adjacent regions of a given map distinct, and the application of Euler's and like theorems to the physical configuration of a country. The known extension of these theories to the features upon a multiply connected closed surface is not given: *e.g.*, upon a Riemann surface.

Of the games and puzzles of position that follow, some classes admit of further and perhaps indefinite development. About the best of these is concerned with arranging m trees in n rows of p trees in a row, m being as small, and n, p as large as may be. Others are ferry-boat and shunting problems,—the latter closely allied to the "15 puzzle" which is given later,—and the study of tesselation, here barely mentioned but which might be made very interesting if connected with questions of symmetrical coloring in $m^2 + n^2$, $2(m^2+mn+n^2)$, or $m^2 + mn + n^2$ colors, according as the primary form used was the square, the equilateral triangle, or the hexagon.

Reiss's theory of the solitaire-board * is referred to, and its leading idea might well have been given, for it is pretty and not difficult; moreover, this might suggest a like theory for boards with rows intersecting at 60°.

In subsequent chapters are treated problems connected with cards, the "fifteen puzzle," the tower of Hanoï, the Chinese rings, the n queens problem, Kirkman's arrangements of school-girls, magic squares; also unicursal problems, including those of mazes, of Hamilton's "game," and of the knight's path. Most of these ingenious puzzles have instructive theories here admirably stated, or are valuable as studies of symmetry.

Of the n queens problem,—to place n chess-queens on a board n cells square so that no queen can take another,—all the "fundamental solutions" are given up to $n = 8$ inclusive. The statement (p. 88) that, whenever $n = 4m + 2$, a solution $2\,4\,6 \ldots n\ 1\,3\,5 \ldots (n-1)$ exists — *i. e.*,

* *Crelle's Journal*, vol. 54, 344, or *Lucas*, vol. 1, 89.

one wherein the columns 1, 2, 3, ... correspond respectively to the rows 2, 4, 6, ... — is not true when n has the factor 3. Two variations of this problem will suggest themselves: the tangents of the direction-angles of the queen's lines of action, viz. 0, ∞, ± 1, may be replaced by $\pm \frac{1}{2}$, ± 2 as in the knight's move, or by other simple ratios of integers; and secondly, whether the directions be thus replaced or not, the board may be made self-returning vertically and horizontally, so that next after row or column n comes 1 again,—a change that allies the problem to the elementary theory of numbers.

With the usual queen's move, every solution for a horizontally self-returning board will hold for a board that is also vertically self-returning; and every such solution will give another, or be reproduced, not only when the board is revolved or reversed, but when we cyclically permute either rows or columns in the order 1 2 3 . . . . n, and likewise when we cyclically permute rows and columns alike in the order 1, $1 + r$, $1 + 2r$, . . . wherein r is prime to n. I find that, for $n < 12$, all such solutions are in these ways reducible to three fundamental ones: 1 3 5 2 4, 1 3 5 7 2 4 6, 1 3 5 7 9 11 2 4 6 8 10, the last of which becomes 1 4 7 10 2 5 8 11 3 6 9 when the cyclic permutation 1 4 7 . . . . is applied to both rows and columns. Solutions, having like these four the forms 1 3 5 . . . . and 1 4 7 . . . , exist whenever n is prime to 6; but the question remains whether, for $n > 11$, solutions not of the form 1, $1 + s$, $1 + 2s$, . . . also exist.

Some of the school-girl and unicursal problems have a general mathematical interest; and the clear and rather full treatment of them will abundantly repay perusal, and leaves apparently little to criticise. The same is true of Magic Squares, chapter v.; though perhaps a little more might well have been said of Nasik squares and cubes, and of the property that the sum for every four adjacent cells of the self-returning board shall be $2n^2 + 2$. How far can corresponding properties be secured on self-returning boards when the sub-squares considered have (say) 9 cells each instead of 4, n being odd, or when instead of rows, columns and diagonals, we consider lines parallel, for instance, to a chess-knight's moves? Among the few *corrigenda* to be made, is this: p. 116, l. 22, and p. 117, l. 9, read " . . . in any order whatever that puts complementary numbers in complementary positions."

Chapter III., on "Some Mechanical Questions," is by far the most open to criticism, and must have been written or compiled in haste; but the "questions" themselves are instructive, and so are some of the oversights.

After an excellent refutation of the paradox of Achilles and the tortoise, and a rather less happy reply to the paradox

that nothing can move, since nothing can move where it is nor yet where it is not, we are shown how an ice-boat, gliding without friction or lee-way, may sail faster than the wind. The formula given for the boat's velocity is right, and is deducible from the condition that the wind's motion relative to the sail be along the sail; but the pressure is wrongly taken to vary generally as the normal component of the wind's motion relative to the sail. Probably for a very broad low sail, the pressure varies nearly as the square of that component; while, for a very narrow high sail, Rayleigh's formula (*Phil. Mag.*, Dec. 1876) would give

$$p = 0.005\, v'^2\, (4 + \pi) \sin \varphi \,/\, (4 + \pi \sin \varphi),$$

wherein v' is the wind's relative velocity in miles per hour, φ the angle its direction makes with the sail, and p the normal pressure in pounds per square foot; and for a plane sail of ordinary dimensions, the law would be intermediate in character.

Next is discussed a curious kind of boat-racing where the crew simply jerk a rope tied to the after thwart. "The effect of each jerk is to compress the boat. Left to itself the boat tends to resume its original shape, but the resistance to the motion through the water of the stern is much greater than that of the bow and hence on the whole the motion is forwards." Doubtless the different shaping of bow and stern does help the effect; but any sensible distortion of the boat seems impossible, neither would it much help matters after all. But if, with each jerk, the crew's center of gravity moves toward the bow m times more rapidly than it moves back on the return, then all is explained; for the boat (supposed for simplicity not to be already under headway) is thrown backward nearly m times as rapidly as forward, causing a water-pressure at the stern at least m^2 times as great as was that at the bow, though lasting only $1/m$ as long.

Then come philosophical remarks on force, inertia, centrifugal force, work and equilibrium; all good enough but for occasional bits of carelessness. Thus (p. 61) we are told that, in throwing an irregular body or in rifling a gun, one tries to give the projectile a rotation about a principal axis; but as no reference has been made to the action of the air, the reader is left to infer that the rotation somehow guides the projectile under the laws of inertia. Then, after an anecdote from which perhaps the author's political sympathies may be surmised, comes Montucla's estimate of the time Archimedes must take in order to "move the world" one inch: over 3×10^{18} centuries. It would indeed take some such time to *raise* by one inch, against terrestrial gravity, a mass equal to

the earth's; but nothing has been said of gravitation, only of mass, inertia, etc., and using Montucla's data we know that to *move* the earth one inch, just as gravitation moves a falling body from rest, would have taken Archimedes only about six weeks.

Finally, after some good pages on gravitation-toys, perpetual motion, and the want of stiffness in large models as compared with small, we come to "Motion of Fluids and Motion in Fluids." Here the author, making use of Hauksbee's empirical law that "when a fluid is in rapid motion, the pressure is less than when it is at rest," forgets that the diminished pressure observed is not the effect but only the cause of the more rapid motion. The fluid is accelerated in flowing from a region of greater pressure to one of less; but a forced acceleration by bellows or blast-fan may not diminish the pressure. Were it right to reason as if mere concomitance among phenomena were causation, we should often find the Hauksbee effect offset, or more, by transmitted pressure from the fan, etc. Thus it is not true (p. 71) that, when we blow through a tube made by placing a paper on two books laid parallel, "the paper will be sucked in," unless the tube be larger than the entering current; nor that a "disk will be drawn towards" a simple cylindrical tube that blows squarely upon it. If the tube flare, or have something like another disk at its mouth, the effect named comes from a partial vacuum between the flaring sides or the two disks. In the pretty experiment of blowing "close and parallel" to a card (p. 72), the card's motion toward the blast is due to the lateral inrush of air to replace what has mingled and been carried off with the blast.

What suggested the law to Hauksbee (see *Phil. Trans.* 1704, p. 147) was the low barometer during a certain hurricane; and he adds the shrewd remark, which with our better means of observation might be followed up, that "on extraordinary gusts a visible vibration of the quicksilver appeared." [Does the "vibration" precede or follow the gust, and does it begin with a rise or a fall?]—Yet even here we see how unsafe a reed, to lean upon, this law would be; for though on the whole the barometer does commonly fall as the wind for a while rises, while the storm-centre is approaching, yet at the very centre the lowest pressure of all is most apt to accompany a brief calm.

To explain the deviation of a curved ball in the air on the principle that where the motion is greatest the pressure must be least, the author has to assume that the path curves toward that side of the ball whose motion is forward by virtue of rotation as well as of translation; but tennis-players tell us the reverse, and from the following experiment it seems that they

are right. Hang a tennis-ball by a doubled string from the ceiling, twist the string, and swing the pendulum. If the string, in untwisting, spins the ball counter-clockwise, then the plane of oscillation slowly shifts counter-clockwise, showing that the right-hand side of the ball's path is convex; and when, presently, the ball begins to revolve the other way, so does the plane of oscillation. The result is the same, if a vertical wall be parallel and close to the plane of vibration. I do not know how completely the true theory of a curved ball has been worked out. It is said, and I suppose rightly, that when the ball, spinning counter-clockwise, flies northward, the friction heaps up a mass of air N.E. of it which travels with it and takes largely the brunt of the resistance to the ball's flight, and whose base giving way to the S.E. and S.W. pushes the ball to the west. Further, I conceive that the atmosphere carried along by the ball and spinning with it is gradually changed by diffusion and by the wind of translation, being mainly collected on the north side and blown off on the south side, carrying with it its eastward motion while the ball recoils to the west. The two actions are somewhat connected; but were the ball replaced by a long broad band between two small pulleys, the first action would be the chief if, in their flight, the pulleys followed one another, and possibly the second if they moved abreast.

It is stated (p. 71) that, "in a perfect elastic fluid . . . . $p = \Pi\alpha^{-v^2}$" where p, v, Π, α denote the pressure, the velocity, and constants. This is true only if γ, the ratio of the specific heat under constant pressure to that under constant volume, be 1; but for every known gas γ exceeds 1, and it must do so theoretically; indeed, for what is called sometimes the perfect gas of theory, $\gamma = 1.67$, while for the more perfect of actual gases, as hydrogen, $\gamma = 1.42$.

The chapter closes with the "unexplained" fact that some large birds follow a ship rapidly and long with no perceptible motion of the wings. It has been suggested, as a possibility, that they utilize air-eddies made by the ship, keeping in the up-draught. By Rayleigh's formula this might be, if the effective wing-surface were equivalent, by reason of its good shape, to 3 square feet of Rayleigh's resisting plane per pound of the bird's weight, and if the bird's horizontal resistance were negligible, his velocity relative to the wind 15 miles per hour, and the upward current 3 miles; but these assumptions may appear extravagant. It is known that, to a less extent, any differences of horizontal air-currents could also be utilized. I was once told by a seaman, I know not how accurately, that he had seen albatrosses (?) hover over the deck, apparently motionless against a stiff breeze, but really with a rapid wing-tremor of very small amplitude. This would help:

the overlapping feathers would simply open as they rose, and close up as they descended.*

The last eighty pages, entitled "Mathematical Problems and Speculations," form almost a separate work. Beside brief and clear sketches of successive attempts to double the cube, trisect the angle and square the circle; of astrology; and of calendars and the ordinary measurement of time—they take up two topics of great philosophical interest: hyperspace, and the constitution of matter.

By "hyperspace" is meant not only the space of n dimensions but also the so-called "curved space" of pan-geometry. The author holds that the hypothesis of our 4-dimensional existence helps to clear up some difficulties in the conception of the ether, in the explanation of gravity, in the supposed fact that the number of kinds of matter is finite and all atoms of a kind are alike, and in the approximate periodicity of atomic weights and other properties of the elements. To our space and to all bodies in the universe, he gives a uniform thickness in the fourth dimension, so that in that dimension they rest on, or in, a homogeneous thin elastic "medium," which is either a body or simply a hyperspace, and by whose vibrations all force and all radiant energy are transmitted. This medium must be thin in the fourth direction, since these energies are as the inverse square and not the inverse cube, etc., of the distance, unless the distance be very small; but its thickness has the same order of magnitude as atomic or molecular diameters, so that atomic forces do not follow the law of the inverse square. It secures that the number of modes of atomic vibration be finite, and hence also the number of kinds of matter,—"as two clocks whose rates are nearly the same tend to go alike if their cases are connected." Finally, if the atoms were formed by some simple or complex harmonic disturbance, the formation might well occur only at a particular phase or phases; and atoms so formed might naturally run in series, as in Mendeléef's periodic law. At least, so runs the dream.

These suggestions appear to me the finest thing in the book. They are but little developed, and not always clearly: *e. g.*, passages on page 197 may seem to imply, contrary to what we know of polarization, that the transverse luminiferous vibrations are in the fourth dimension.† Nevertheless the possibilities of the proposed theory may be immense. Of course all depends upon the success with which it can be applied not only to philosophical generalities, but to the definite predic-

* See *Nature*, vols. 8, 9, 23, 27–29, 35–37, 40, 42–44; and *Science*, vols. 10, 11, 13, 17.

† See also *Messenger of Mathematics*, XXI., pp. 20–24.

tion of laws; still, *a priori*, the hypothesis would seem to be quite admissible. Certain details of the theory remain to be fixed as may be most advantageous. For instance, the medium may have either one or more dimensions besides the familiar three, and its extent in the new directions is as yet undetermined. It may be self-returning in the new directions, instead of ending abruptly; and so on. It may even *shade off* in the new directions, the materials of the world tending toward our three-fold space as a region of maximum density; either through some kind of selection such as Mr. C. S. Peirce has suggested, or of quasi-attraction, a little as in Hinton's *Scientific Romances*. Such shading would follow some rapid exponential law, as in the theory of distribution of errors; and its modulus, however small, would probably be definite.

The useful assumption "that the measure of distance remains the same everywhere" (p. 201), does not necessarily imply that every so-called "curved space" lies in an uncurved space of more dimensions, but only that the relation among the mutual distances of four quasi-complanar points be always that known to hold among the geodetic distances apart of four points on a sphere whose radius is either real or purely imaginary. Thus it is not true that a two-dimensioned pseudospherical space must be finite unless "constructed in space of four dimensions" (pp. 200–1): and the oversight, made by so good a writer, goes to justify Klein's criticism that the phrase "curved space" is misleading. Why not rather, in the general case, describe the space as *quasi-curved?*

The rest of the account given of hyperspace and of the constitution of matter is well thought out and clear. It will help the general reader toward some truthful notions as to studies which may perhaps play an important part in the near future.

J. E. OLIVER.

ITHACA, N. Y., *November* 15, 1892.

RECENT STAR CATALOGUES.

Zweites Münchener Sternverzeichniss, enthaltend die mittleren Oerter von 13200 *Sternen, für das Aequinoctium*, 1880. Beobachtet und berechnet von Dr. JULIUS BAUSCHINGER, Observator der Sternwarte. Munich, 1891. 4to, pp. xxvi. and 172.

THE Scottish astronomer, John Lamont, long director of the observatory at Bogenhausen, a suburb of Munich, had caused the observation, about the middle of this century, of about 33,000 stars; they were taken in zones, and by an ingenious method for saving labor. Unfortunately, the assist-

ants, who made these observations, had fallen into a good many errors, and a thorough revision was necessary. This was begun by Feldkirchner, then the principal assistant at the observatory; and the corrected catalogue was published some years ago as vol. I. of the *Neue Annalen*, by Prof. Seeliger, Lamont's successor. In this revision it was found necessary to reobserve about 13,000 stars, those, namely, of the 33,000 which were not sufficiently controlled by duplicate observations and other catalogues. All these were observed, usually once, but occasionally twice or more, by Dr. Bauschinger, helped in reading the microscopes by Mr. List. The instrument was the old Reichenbach circle of the observatory, with an aperture of 11 centimetres; it has been provided with microscopic reading in place of Reichenbach's verniers.

The observations were accomplished between 1884 and 1888; stars were added to employ the time where large gaps existed in Lamont's zones; and the results have been published in the form of a catalogue for the epoch 1880. Many of the 13,000 stars are faint ones, picked up by Lamont's assistants on especially clear nights, and therefore less likely to be duplicated at Bogenhausen or elsewhere; while their faintness rendered it difficult for Bauschinger to see them on average nights. These especially difficult stars seem to be observed with probable errors of $\pm 0^s.14$ and $\pm 1''.3$ in right ascension and declination respectively. These numbers are not too large for the circumstances. Dr. Bauschinger estimates the probable error of a zone-star observed under average conditions at $\pm 0^s.081$ in right ascension, and $\pm 0''.85$ in declination; while a comparison made by myself, but including only stars to $8^m.2$ inclusive, gave probable errors about $\frac{1}{10}$ less in amount. The limit of magnitude corresponds nearly to that adopted by Argelander at Bonn, with an instrument a little larger, as the boundary of his classification of "brighter" stars, such as can be readily seen under average circumstances with full field illumination. The same limit, modified to suit an aperture of 12·2 centimetres, I have found to hold good for the Repsold circle at Williamstown; stars below $8^m.5$ require less field illumination than I habitually use unchanged for brighter ones. Dr. Bauschinger's work there is entirely adequate to its purpose. This would not, in fact, require this degree of accuracy; but it is also a source of star-positions, which are useful in many other ways. In a paper later in the same volume, Dr. Bauschinger discusses the proper motions of 90 stars. These are such as were detected in comparing Lamont's and other catalogues. After a proper motion was thus made highly probable other existing catalogues were searched for data, and, after bringing everything up to 1880, a least square discussion was undertaken with partial allowance for system-

atic corrections. The weights here assigned are rather a rude approximation; this is pardonable where the object is mainly to make sure of the existence and general direction of the proper motion. But when the purpose in view is to obtain the most probable values of the motions, the systematic corrections and the weights should be rather more strictly applied.

Lalande's zones, for example, usually deserve no more than about $\frac{1}{30}$ of the weight of the Pulkova catalogue of 1855, while here we find $\frac{1}{4}$ employed instead. But this secondary correction is, as above stated, here of quite subordinate importance.

T. H. SAFFORD.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, November 5, at half-past three o'clock, the president in the chair. The following persons, having been duly nominated, and being recommended by the council, were elected to membership: Professor Cleveland Abbe, U. S. Weather Bureau; Professor Henry S. White, Northwestern University; Mr. Gardner Ladd Plumley, Home Life Insurance Company of New York. It was announced that the president and secretary had been made members of the international committee on the proposed joint memorial to Gauss and Weber at Göttingen. No formal papers having been announced, a general discussion upon mathematical topics was in order. Miss Williams made some remarks, in which she stated that Steiner's method of proving, that the circle has a greater area than any other plane figure having an equal perimeter, appeared to lack rigor, because he took for granted that a maximum exists, and the latter proposition had not been demonstrated. Edler has found a rigorous proof of this proposition relating to the circle; but for the corresponding proposition with reference to the sphere, no rigorous proof by elementary geometrical methods seems to be known. Steiner himself objects to a similar method of proof, which Lhuilier applies to triangles. After stating his objection, Steiner gives a concise and rigorous proof of this proposition. When he goes on to the circle, he seems to forget his objection and to adopt a somewhat similar method. Most of the discussion which followed Miss Williams's remarks turned on this point. Dr. Fiske made some general remarks upon a recent paper* treating of the mathematical theory of

* Mathematical Investigations in the Theory of Value and Prices, by Dr. Irving Fisher, Yale University. *Transactions of the Connecticut Academy*, vol. IX., July, 1892.

value and prices. Professor Pupin presented a preliminary note on the analytical properties of the kinetic energy function in a system for which the principles of least action and of conservation of energy are both true.

THE University of Padua announces a *fête* for December 7, 1892, in honor of the three-hundredth anniversary of the day when Galileo became a professor in that institution.

THE senate of the University of Cambridge recently discussed a proposal to establish a mechanical sciences tripos for engineering students. The result was unexpectedly favorable to the scheme, and it is probable that the new tripos will meet with no opposition from the more mathematical members of the council who at first objected to it.

DR. C. H. CHAPMAN, recently of the department of mathematics at Johns Hopkins University, has been appointed conductor of the Wisconsin Teachers' Institute at Milwaukee.

T. S. F.

A CIRCULAR has been issued asking for subscriptions to aid in striking a portrait medal of M. Hermite, to be presented to him in commemoration of his approaching seventieth birthday. Professor Simon Newcomb is a member of the committee having the matter in charge.

DR. ARTHUR A. RAMBAUT has been elected royal astronomer of Ireland, and professor of astronomy in the University of Dublin. He succeeds Sir R. S. Ball, now at Cambridge.

WE have to record the death of Robert Grant, F.R.S., late professor of astronomy at the University of Glasgow, and that of Dr. Löwenherz, director of the Imperial Physical Institute of Berlin.

H. J.

AMONG the recent announcements of B. G. Teubner, of Leipzig, the theory of differential equations and the closely allied subject of continuous groups of transformations are most prominently represented. Thus, in addition to the third and last part of Sophus Lie's *Theorie der Transformationsgruppen*, edited by Dr. F. Engel, we are promised in 1892 the same author's *Vorlesungen über continuierliche Gruppen, zur Einführung in die Theorie derselben*, edited by Dr. G. Scheffers, and at least the first of the two volumes of Dr. L. Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, while for 1893 a work by Dr. L. Heffter, *Einleitung in die Theorie der gewöhnlichen linearen Differentialgleichungen*, is in preparation, as well as a German translation,

by H. Maser, of A. R. Forsyth's *Theory of differential equations*, part I.

The publication of the works of the late L. Kronecker will begin in 1893. The first volume will contain Kronecker's lectures on determinants (never before printed), edited by Dr. K. Hensel; this is to be followed by a volume comprising the collected papers of Prof. Kronecker on the theory of determinants and algebraic forms. At the same time Dr. E. Netto has undertaken the edition of Kronecker's lectures on definite integrals. A work by O. Stolz on the infinitesimal calculus, *Grundzüge der Differential- und unbestimmten Integralrechnung*, will be welcomed in particular by those acquainted with the same author's excellent *Vorlesungen über allgemeine Arithmetik* (1885 and 1886). Like this latter work the new one will be divided into two parts, one being devoted to real functions with real variables, the other to complex functions of real and complex variables. The first part is to appear in 1893. P. Bachmann, *Elemente der Zahlentheorie;* H. Stahl and V. Kommerell, *Die Grundformeln der allgemeinen Flächentheorie;* and R. Sturm, *Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung*, part II., *Die Congruenzen erster und zweiter Ordnung*, are other announcements for 1892 by the same house.

On applied mathematics we may mention C. Neumann's *Beiträge zu einzelnen Theilen der mathematischen Physik*, and the continuation of Steinheil and Voit's *Handbuch der angewandten Optik*, of which the second volume is promised for 1893, the third for 1894. A. Z.

At the annual meeting of the London Mathematical Society, which was held on November 10, the following officers were elected for the coming year: president, A. B. Kempe; vice-presidents, A. B. Basset, E. B. Elliott, and A. G. Greenhill; treasurer, J. Larmor; secretaries, M. Jenkins and R. Tucker; other members of the council, H. F. Baker, A. R. Forsyth, J. W. L. Glaisher, J. Hammond, M. J. M. Hill, E. W. Hobson, A. E. H. Love, Major Macmahon, and J. J. Walker. The address of the retiring president, Professor A. G. Greenhill, had for its subject collaboration in mathematics.

During the present term the following advanced courses in pure mathematics are being given at Johns Hopkins University,—by Professor Craig: partial differential equations, twice a week; linear differential equations, three times a week; theory of functions and mathematical seminary, twice a week; by Professor Franklin: modern algebra and higher plane curves, five times a week; determinants, theory of equations, and advanced analytical geometry, three times a

week; calculus (special topics), twice a week; by Mr. Hurlburt: theory of substitutions, three times a week.

DR. WILLIAM H. METZLER has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

DR. ASAPH HALL, Jr., recently of the U. S. Naval Observatory, has been appointed director of the "Detroit Observatory" and professor of astronomy at the University of Michigan, Ann Arbor. T. S. F.

MR. GEORGE P. METZLER, Ph.D. (Johns Hopkins), lately of Clark University, has been appointed instructor of mathematics at the University of Michigan.

IT will be interesting to our readers to learn that the lectures delivered by Professor Felix Klein, at the University of Göttingen, in the course of the last three years, have been lithographed, and can be obtained by applying to Mr. Fr. Schilling, Cand. math., Nikolausberger Weg 19, Göttingen. The whole set consists of six parts, covering as many semesters, viz., 1889-90, *Nicht-Euklidische Geometrie* (Mk. 11.50); 1890-91, *Ausgewählte Kapitel aus der Theorie der linearen Differentialgleichungen zweiter Ordnung* (Mk. 11.50); 1891-92, *Riemann'sche Flächen* (Mk. 10.00). A. Z.

ERRATA in article "On the Non-Euclidian Geometry," in the BULLETIN for November: p. 21, l. 23, for "lectures" read "lecture"; p. 25, l. 32, for k^2 read k^{-2}; p. 27, last line, also p. 28, l. 5 from bottom, for r^2 read r^{-2}; p. 29, line 15, for a read b; p. 33, footnote, for "Richard" read "R. S."

E. M.

THE committee of publication begs to announce that, beginning with the present number, it will enjoy the coöperation of Professor Alexander Ziwet in the editorial work of the BULLETIN.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

ABHANDLUNGEN zur Geschichte der Mathematik. Heft vi. Leipzig, Teubner, 1892. 8vo. 204 pp. 1 plate. M. 5.00

AMBROSINO (V.). Sulla determinazione della longitudine di un luogo col metodo delle distanze lunare. Messina 1892. 8vo. 37 pp. 1 plate. M. 1.50

AUTOMARI (X.). Leçons de cinématique et de dynamique, suivies de la détermination des centres de gravité. Paris 1892. 8vo. 4 and 233 pp. fig.

BASSET (A. B.). A treatise on physical optics. London, Bell, 1892. 8vo. 430 pp. 16s.

BERLET (B.). Adam Riese, sein Leben, seine Rechenbücher und seine Art zu rechnen. Die Coss von Adam Riese. Frankfurt a. M. 1892. 8vo. 64 pp. Portrait, facsim. M. 1.00

BIGELOW (F. H.). Notes on a new method for the discussion of magnetic observations. Washington, Weather Bureau, 1892. 8vo. 40 pp. 1 plate.

BINDER (W.). Neue Axenconstructionen eines durch fünf beliebige Bedingungen gegebenen Kegelschnittes nach projectivischer Methode. Wien 1892. 8vo. 21 pp. fig. M. 1.20

BLAIKIE (J.) and THOMSON (W.). A text-book of geometrical deductions. Books 1 and 2, corresponding to Euclid, Books 1 and 2. Fcap., 208 pp. Longmans. 2s. 6d.

BLAKESLEY (T. H.). Les courants alternatifs d'électricité. Traduit de la 3me édition anglaise et augmenté d'un appendice par W. C. Rechniewski. Paris 1892. 12mo. fig. M. 6.50

BREMIKER (C.). Tavole logaritmico-trigonometriche con 5 decimali. 4ta edizione, eseguita per cura di L. Cremona. Milano 1892. 8vo. 196 pp. M. 1.80

BRISSE (Ch.). Cours de mécanique à l'usage de la classe de mathématiques spéciales, entièrement conforme au dernier programme d'admission à l'École polytechnique. Paris, Gauthier-Villars, 1892. 8vo. fig. Fr. 3.25

BURN (R. S.). Mechanics and mechanism, being elementary examples of the leading principles of mechanics, together with practical diagrams and drawings of the principal mechanical movements. 7th ed., revised and enlarged. London, Ward, 1892. 8vo. 150 pp. 200 engravings. Cloth. 2s. 6d.

DEMARTRES. Cours d'analyse professé à la faculté des sciences de Lille 1890–91 et rédigé par E. Lemaire. Partie II: Propriétés des fonctions analytiques. Paris 1892. 4to. 172 pp. fig. (Lith.) M. 9.00

DICKSTEIN (S.). Zasady teoryi liczb etc. (On the foundations of Wronski's theory of numbers.) Kraków 1892. 8vo. 122 pp. (Polish.) M. 4.00

DZIOBEK (O.). Mathematical theories of planetary motions. Translated by M. W. Harrington and W. J. Hussey. Ann Arbor, Inland Press, 1892. 8vo. 8 and 294 pp. Cloth. $3.50

EBERLE (J. F.). Ueber rationale Curven 5. Ordnung, insbesondere diejenigen 4. und 5. Classe. München 1892. gr. 8. 34 pg. m. 1 Tafel. M. 1.00

ÉCOLE NAVALE. Types de calculs nautiques. Paris 1892. 4to. 124 pp. fig. Fr. 5.00

ÉLIE (B.). La fonction vectorielle et ses applications à la physique. Bordeaux 1892. 8vo. 124 pp. Fr. 7.50

ERLER. Die Elemente der Kegelschnitte in synthetischer Behandlung. Zum Gebrauche in der Gymnasialprima. 4te verbesserte Auflage. Leipzig, Teubner, 1892. 8vo. 49 pp. 1 plate. M. 1.20

FISCHER (J.). Berechnung und Wirkungsweise der elektrischen Gleichstrommaschinen. Zürich 1892. 8vo. fig. M. 4.20

FLEMING (J. A.). The alternate current transformer in theory and practice. Vol. II : The utilisation of induced currents. London, "Electrician" Office, 1892. 8vo. 612 pp. 12s. 6d.

FONTENÉ (G.). L'hyperespace à n-1 dimensions. Propriétés métriques de la corrélation générale. Paris, Gauthier-Villars, 1892. 8vo. fig. Fr. 5.00

GAUSS (F. G.). Die trigonometrischen und polygonometrischen Rechnungen in der Feldmesskunst. 2te Auflage. (In 9 Heften.) Halle 1892. 8vo. Heft 1–4, pp. 1–320, fig. Mk. 14.00

GEF (W.). Die Wärmequelle der Gestirne in mechanischem Maass : ein Beitrag zur mechanischen Wärmetheorie. Heidelberg 1892. 8vo. 11 pp. M. 1.00

GRAVELAAR (N. L. W.). Leerboek der Planimetri. Groningen 1892. 8vo. 12 and 182 pp. M. 2.50

GUSTAWICZ (B.). Theory of the loxodrome and the loxodromic triangle, with reference to the construction of sea-charts and the solution of nautical problems. (Polish.) Part I, 1891 : Part II, 1892. Kraków. 8vo. 45 and 49 pp. M. 3.60

HALL (H. S.) and KNIGHT (S. R.). Algebra for beginners. (Without Answers.) Fcap., pp. 196. Macmillan. 2s.

HALLERSTEIN (H. v.) Lehrbuch der Elementar-Mathematik. Herausgegeben und erweitert von B. Hülsen. Theil 1 : Arithmetik. Berlin 1892. 8vo. M. 4.20

HEGELE (A.). Die Fundamentalaufgaben der darstellenden Geometrie. Straubing 1892. 8vo. 31 pp. 16 plates. M. 2.00

HOBSON (E. W.) and JESSOP (C. M.). An elementary treatise on plane trigonometry. Cambridge, University Press (Macmillan), 1892. 12mo. 300 pp. Cloth. 4s. 6d.

HOSKINS (L. M.). The elements of graphical statics. A text-book for students of engineering. New York, Macmillan, 1892. $2.25

HOUEL (J.). Tables de logarithmes à cinq décimales pour les nombres et les lignes trigonométriques, suivies des logarithmes d'addition et de soustraction ou logarithmes de Gauss et de diverses tables usuelles. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1892. 8vo. 48 and 49 pp. Fr. 2.00

JARMAN (J. A.). Algebraic factors, classified and applied. London, Macmillan, 1892. 8vo, 150 pp. 2s.

KAIL (J. A.). Das ebene Dreieck als specieller Fall des sphärischen. Wien 1892. 8vo. 16 pp. M. 1.00

KLEIN (F.). Vorlesungen über die Theorie der elliptischen Modulfunctionen, ausgearbeitet und vervollständigt von R. Fricke. Vol. II: Fortbildung und Anwendung der Theorie. Leipzig, Teubner, 1892. 8vo. 16 and 712 pp. fig. M. 24.00

KLEKLER (K.). Die stereographische Projection als Hülfsmittel der ebenen Darstellung sphärischer Constructionen. Wien 1892. 8vo. 20 pp. 3 plates. M. 1.50

KRAZER (A.) und PRYM (F.). Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen. Leipzig, Teubner, 1892. 4to. 12 and 133 pp. M. 7.20

LAGRANGE (J. L.). Oeuvres complètes, publiées par les soins de J. A. Serret et G. Darboux, sous les auspices du ministère de l'instruction publique. Tome XIV. et dernier: Correspondance avec divers savants, publiée et annotée par L. Lalanne. Paris, Gauthier-Villars, 1892. 4to, 12 and 849 pp., 2 facsim. Fr. 15.00

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ON PETERS'S FORMULA FOR PROBABLE ERROR.*

BY PROF. W. WOOLSEY JOHNSON.

THERE are three quantities which may be used as measures of the risk of error in an observation, each of which, according to the law of facility, bears a fixed ratio to the reciprocal of the measure of precision; namely, the probable error r, the mean error ε, and the mean absolute error η. Their values, in terms of h, namely

$$r = \frac{\rho}{h}, \qquad \varepsilon = \frac{1}{h\sqrt{2}}, \qquad \eta = \frac{1}{h\sqrt{\pi}},$$

may be called their *theoretic values*. By definition these would, in an infinite number of observations of the same precision, be respectively: the error which in order of absolute magnitude stands in the middle of the series, that whose square is the mean of the squares of the errors, and the mean of the absolute values of the errors.

The quantities similarly defined with reference to a series of n given observations may be called the *observational values* of r, ε and η. The assumption of the equality of the theoretic and observational value of either of the measures of the risk of error will assign a value to h and to each of the other measures. With respect to r, such an assumption would obviously be unsatisfactory, except when n is very large; but with respect to ε and η, whose observational values are

$$\sqrt{\frac{\Sigma e^2}{n}} \quad \text{and} \quad \frac{\Sigma [e]}{n},$$

the assumptions give the methods of determining h and r which are actually in use. Thus the ε-method gives

$$r = \rho\sqrt{2}\sqrt{\frac{\Sigma e^2}{n}} = .6745\sqrt{\frac{\Sigma e^2}{n}}\,; \quad . \quad . \quad . \quad (1)$$

and the η-method gives

$$r = \rho\sqrt{\pi}\,\frac{\Sigma [e]}{n} = .8453\,\frac{\Sigma [e]}{n}. \quad . \quad . \quad . \quad (2)$$

The first formula is preferred because we can prove otherwise that the corresponding value of h is the most probable

* Abstract of a Paper read before the Society, June 4, 1892.

value, but the second gives very good results, and is much easier of application, especially when n is large.

Each of these formulæ however gives the value of r on the hypothesis that we are in possession of the true value of the magnitude observed: taking the arithmetical mean as the final value, they give the value of r', which we may call the *apparent probable error;* thus the first formula gives

$$r' = \rho \sqrt{2} \sqrt{\frac{\Sigma v^2}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The correction in order to produce r is usually based on the consideration that, since Σv^2 is the minimum value of Σe^2, we shall do the best we can, if we add to it the mean value of the excess $\Sigma e^2 - \Sigma v^2$ due to the unknown error δ in the arithmetical mean. This is shown to be ε^2, so that $\frac{\Sigma e^2}{n}$ is taken to be $\frac{\Sigma v^2}{n-1}$;

and
$$r = \rho \sqrt{2} \sqrt{\frac{\Sigma v^2}{n-1}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To correct the value of the apparent probable error by the η-method, namely

$$r' = \rho \sqrt{\pi} \frac{\Sigma [v]}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

by a similar consideration of mean values would be very difficult, nor would the result be satisfactory. If we construct the value of ε as an ordinate to the supposed "true value" as an abscissa, we shall have a parabola with vertex downward, the minimum ordinate corresponding to the arithmetical mean of the n observed values. The mean value of the ordinate according to the law of probability of δ (the divergence of the true value from the arithmetical mean) is then a plausible value to adopt for ε when δ is unknown. But, if we make a like construction for η, we have a polygonal line with an angle corresponding to each observed value as an abscissa, the minimum value occurring not at the arithmetical mean, but at a point which has an equal number of observed values on each side of it.

The concavity of the polygonal line is upward, and it is true that the mean value of the ordinate, according to the law of probability of δ, somewhat exceeds the value corresponding to $\delta = 0$, but its value depends upon the distribution of the observed values.

Sir George Airy, in his "Theory of Errors of Observation,"

published in 1861, takes for η the mean between the absolute values of the mean of the positive and the mean of the negative errors, and holds that no correction should be made on the ground that this value of η does not change for small values of δ. But this is fallacious, for Airy's value of η changes abruptly when the value of δ passes through the value of any one of the residuals, and the probability that δ should exceed one or more of the residuals is considerable. Moreover, there is no justification whatever for Airy's gratuitous departure from the usual definition of η.

Five years earlier, Dr. C. A. F. Peters had given in the *Astronomische Nachrichten* for 1856, vol. XLIV., pp. 29–32, the formula*

$$r = \rho \sqrt{\pi} \frac{\Sigma [v]}{\sqrt{[n(n-1)]}} \quad . \quad . \quad . \quad . \quad (6)$$

which has been generally accepted (although ignored by Airy), and is known as Peters's formula. Comparing this with equation (5), it is seen to give to the ratio $r : r'$ the value

$$r : r' = \sqrt{n} : \sqrt{(n-1)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is the same as that derived from the correction for mean value in the ε-method,—compare equations (4) and (3). But it seems to have been overlooked by writers on Least Squares that Dr. Peters's method establishes directly the ratio between the real and apparent probable errors, and affords a much more satisfactory method of deriving the usual formula

$$r = \rho \sqrt{2} \sqrt{\frac{\Sigma v^2}{n-1}}$$

from equation (3) than does the method of correction for mean error of Σv^2.

Peters's method of establishing equation (7) is substantially as follows:

Denoting by $e_1, e_2, \ldots e_n$ the true errors whose probable error is r, the error of the arithmetical mean is

---

* Peters states that previously the formula generally used had been

$$r = \rho \sqrt{\pi} \frac{\Sigma [v]}{n-1},$$

"which gives to r a value somewhat too great."

I do not know the origin of this method of modifying formula (5). It may have been due to the feeling that the indeterminate form should be assumed when $n = 1$, as in the case of formula (4).

$$\delta = \frac{e_1 + e_2 + \ldots + e_n}{n};$$

and the divergences from the arithmetical mean, whose probable error is denoted by r', are

$$v_1 = e_1 - \delta = \frac{n-1}{n} e_1 - \frac{1}{n} e_2 - \ldots - \frac{1}{n} e_n,$$

$$v_2 = e_2 - \delta = -\frac{1}{n} e_1 + \frac{n-1}{n} e_2 - \ldots - \frac{1}{n} e_n,$$

etc., etc.

Therefore by the formula for the probable error of a linear function of independent quantities we have, from any one of the above equations,

$$r' = \sqrt{\left[\left(\frac{n-1}{n}\right)^2 r^2 + (n-1)\frac{1}{n^2} r^2\right]} = r\sqrt{\frac{n-1}{n}}.$$

In the proof of Peters's formula given by Chauvenet in his Appendix on Least Squares ("Spherical and Practical Astronomy," vol. ii, p. 497) the above is replaced by a piece of erroneous reasoning. He says: "Each observed N may be supposed to be the result of observing the mean quantity x_0 increased by an observed error v. The probable error of $N = x_0 + v$ is therefore

$$r = \sqrt{(r_0^2 + r'^2)} = \sqrt{\left(\frac{r^2}{n} + r'^2\right)}$$

whence,

$$r = r'\sqrt{\frac{n}{n-1}}$$

This amounts to treating the equation $e_1 = v_1 + \delta$, as if e_1 were made up of *independent* observed parts v_1 and δ. It would be more natural to write $v_1 = e_1 - \delta$, whence we should have, by similar reasoning, the erroneous conclusion $r'^2 = r^2 + r_0^2 = r^2\frac{n+1}{n}$, the error arising from the fact that e_1 and δ are *not* independent; and, if we write with Peters $v_1 = e_1 - \frac{\Sigma e}{n}$, we see the necessity of expanding Σe and expressing v_1 in terms of the independent quantities e_1, e_2, $\ldots$ e_n.

Watson, in his "Theoretical Astronomy," p. 374, copies

Chauvenet. The reason given by Merriman ("Text-Book on the Method of Least Squares," p. 93) for the correction of equation (5) is based on the consideration of the mean value of the excess of $\Sigma[e]$ over $\Sigma[v]$, and assumes that this excess has the same relative value as that arising in the ε-method. But it would rather appear that, while the mean excess in the value of ε (in accordance with the law of probability of δ) happens to agree with the ratio of the true to the apparent value of r as rigorously established by Peters, there is no reason to suppose that this would be the case with regard to the mean excess in the value of η. Moreover, as before remarked, if we could obtain this mean excess, the correction founded upon it would not give so satisfactory a formula as that of Peters.

THE THEORY OF TRANSFORMATION GROUPS.

Theorie der Transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. FRIEDRICH ENGEL bearbeitet von SOPHUS LIE, Professor der Geometrie an der Universität Leipzig. Leipzig, B. G. Teubner, 1888. 8vo, pp. viii + 632.

THERE is probably no other science which presents such different appearances to one who cultivates it and to one who does not, as mathematics. To this person it is ancient, venerable, and complete; a body of dry, irrefutable, unambiguous reasoning. To the mathematician, on the other hand, his science is yet in the purple bloom of vigorous youth, everywhere stretching out after the "attainable but unattained," and full of the excitement of nascent thoughts; its logic is beset with ambiguities, and its analytic processes, like Bunyan's road, have a quagmire on one side and a deep ditch on the other and branch off into innumerable by-paths that end in a wilderness.

Among the most important of the newer ideas in mathematics is that of the *group*. In its nature it is essentially dynamic, involving the notion of operating with one thing upon another. Thus, if x and y be two of the entities of the group we shall derive new entities of the same kind by operating with y upon x and with x upon y. Entities failing of this virtue are by that fact excluded from the group.

The individuals of the group may be finite or infinite in number, but mere population does not suffice to classify them; we must consider whether the entities are separated by finite intervals or whether they succeed each other continuously. For instance, granting that the interval between the condi-

tions A and B is finite, let a transformation T change an object from A to B; then in classifying the group to which T belongs we must notice whether or not there is a continuous succession of conditions between A and B through each of which the object may be passed by the transformations of the group. In this way the ideas of continuous and discontinuous groups arise. Discontinuous groups containing a finite number of individuals are sufficiently familiar to modern readers in the theory of substitutions; but there are also partly discontinuous groups, in which the finite intervals separate, not individuals, but hosts* of individuals, the entities of each host being in continuous succession. All possible transformations of rectangular coördinates in a plane form such a group; for they fall into two distinct categories, those where the new and old axes are taken in the same sense and those where they are not. For the first host we have the analytical expression:

$$x' = x \cos \alpha - y \sin \alpha, \quad y' = x \sin \alpha + y \cos \alpha;$$

while for the second,

$$x' = x \cos \alpha + y \sin \alpha, \quad y' = x \sin \alpha - y \cos \alpha.\dagger$$

The number of transformations in a continuous group is infinite from the nature of the case, but still it is perfectly rational to speak of "finite" and "infinite" continuous groups; in the first case the number of *infinities of transformations* is finite, in the second case it is not. To express these distinctions analytically let the variables $x'_1, \ldots x'_n$ be given as functions of $x_1, \ldots x_n$ by the n solvable equations:

$$1) \qquad x'_i = f_i(x_1 \ldots x_n) \quad [i = 1 \ldots n]$$

These equations represent a transformation between the variables x and x'.‡ It is not necessary to the validity of many results that the f_i be analytic functions of their arguments, though this is assumed to be the case unless the contrary is stated. After saying this, it is unnecessary to add that the word "solvable" is not used above in the sense in which it is used in speaking of algebraic equations; it here means merely (1) that the Jacobian

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

* The word "Schaar" is tentatively translated by "host."
† See p. 7. ‡ See p. 1.

shall not identically vanish, and (2) that when in the region* of a certain point x_i^0 $[i = 1 \ldots n]$ we can expand the functions $f_i(x)$ in convergent series proceeding by positive powers of $x_i - x_i^0$, then so long as x_i remains within that region we can expand x_i in a convergent series proceeding by positive powers of $x'_i - x'^0_i$, where x'^0_i is a certain point. It is unnecessary to add that the transformations

$$x'_i = f_i(x)$$

$$x_i = F_i(x')$$

are inverse to one another, and that when performed in succession they produce the identical transformation; but in this connection we may notice the interesting fact that not all groups contain the identical transformation. In his first investigations† Lie endeavored to prove that all groups contained the identical transformation; but he soon recognized his fallacy, and afterwards (1884) Engel, who was of the greatest assistance to Lie in the preparation of this book, discovered a group which actually did not contain it. This group is represented by the equation

$$x' = ax$$

with the arbitrary parameter a; mod $a < 1$. Lie finally found that the equations of every group may, by changing the parameters, and by the process of analytical propagation, be derived from those of a group containing the identical transformation and all of whose transformations can be arranged in pairs of "inverses."

The parameters play a great part in the theory; it is by varying them continuously that we obtain the individuals of the group. In fact the equations 1) completely represent the transformation group only when written in the form:

$$1) \qquad x'_i = f_i(x_1, \ldots x_n, a_1, \ldots a_r)$$

with the arbitrary parameters put in evidence. When the number of parameters is finite we have a "finite continuous group." By assigning a new set of values to the a_k we obtain another transformation

$$x''_i = f_i(x', b)$$

* For this use of "region" cf. Craig's Linear Differential Equations, where I believe "Bereich" found an English equivalent for the first time in America.

† See p. 165.

and if in these equations we replace the x'_i by their values from 1), we must, by the definition of a group, obtain a third transformation

$$x''_i = f_i(x,c)$$

where the c_k are independent of the variables and depend only upon the a_k and b_k.

To reach a useful definition of an infinite continuous group, Lie remarks that we may differentiate the equations

$$x'_i = f_i(x,a)$$

a sufficient number of times with respect to the x_i, eliminate the parameters, and thus obtain a system of differential equations whose most general system of solutions is represented by the equations

$$x'_i = f_i(x,a) \quad [i = 1 \ldots n].$$

Combining this remark with the definition of a group, we see that a *finite* continuous group is defined by a system of differential equations, such that (1) from any two systems of solutions we can always derive a third, just as above from $x'_i = f_i(x,a)$, and $x''_i = f_i(x,b)$, we derive the equations $x''_i = f_i(x,c)$; and (2) the most general solution contains only a finite number of parameters. Granted now that we have a system of differential equations whose most general solution depends not merely upon a finite number of arbitrary constants but upon arbitrary elements of a higher order, say arbitrary functions, then the host of transformations satisfying this system will form an *infinite* continuous group. For example the equations $\frac{\partial F_i}{\partial x_k} = 0 \quad [i \gtrless k;\ i, k = 1 \ldots n]$ define an infinite continuous group,

$$x'_i = \Pi_i(x_i) \quad [i = 1 \ldots n]$$

the functional symbol Π being clearly perfectly arbitrary.

Upon the foundation of these principles and definitions Lie constructs his theory. The first part of his work, which is now under consideration, is devoted to questions of an entirely general character; the derivation of the fundamental differential equations, the infinitesimal transformations, systems of equations which admit infinitesimal transformations, determination of all the sub-groups of a given group, and other problems of wide generality. The second part treats of the "contact transformations" and will not be specially con-

sidered in the present article. Lie's* "Elementarlehrbuch" on differential equations is devoted to ordinary equations of the first, second and third orders and to linear partial ones in three and four variables, which admit known infinitesimal transformations. He shows that those general classes of differential equations which were integrated by the older mathematicians can be characterized as the most general differential equations admitting certain groups. Owing to its elementary character and to its numerous examples it is likely that Lie's theories can be most easily approached through this work by a person entirely unacquainted with them. Reserving, for the present, further remarks upon this most interesting volume, we return to the general theory as set forth in part I.

The literary style of the book deserves mention on account of its perfect simplicity and clearness; the author has carefully avoided adding to the difficulties of the subject by obscure and involved sentences. A patient and persistent effort to make things understood, so far as careful statements, frequent *résumés*, and abundant illustration can do it, is felt throughout the volume.

The r parameters $a_1, \ldots a_r$ contained in the equations of the group may or may not all be essential. To determine this point let the $f_i(x, a)$ be expanded in powers of $x_i - x_i^0$ $[i = 1 \ldots n]$; the coefficients in the series are analytical functions of $a_1, \ldots a_r$, and if r of them are independent, then clearly by giving to $a_1, \ldots a_r$ all possible values we shall obtain ∞^r different transformations. In this case the parameters are all essential; but it is possible that no r of the coefficients may be independent, all of them being expressible in terms of a less number. We can not then obtain ∞^r transformations by varying the $a_1, \ldots a_r$ and may, by introducing functions of the $a_1, \ldots a_r$, diminish by at least unity the number of parameters in evidence. Let the new parameters be $A'_1, \ldots A'_{r-m}$. We can clearly construct at least one linear partial differential equation

$$2) \qquad \sum_1^r{}_k \, X_k(a_1 \ldots a_r) \frac{\partial f}{\partial a_k} = 0$$

which shall be identically satisfied by $A'_1, \ldots A'_{r-m}$; it will also be satisfied by $f_1, \ldots f_n$ since $f_1, \ldots f_n$ regarded as functions of the a_k depend only on $A'_1, \ldots A'_{r-m}$. Conversely, if the $f_1, \ldots f_n$ do satisfy one or more equations of the type 2) the parameters are not all essential. This characteristic property of the functions $f_1, \ldots f_n$ is of course of

* This book was arranged and prepared for publication by Scheffers.

enormous importance in the theory, since the number of independent infinitesimal transformations of a group depends upon the number of its essential parameters.*

The parameters being all essential, we now observe that within the scope† of all values of the a_k and the x_i $[k = 1 \ldots r,\ i = 1 \ldots n]$ we may choose regions (a) and (x) of such a nature that within both regions the $f_1, \ldots f_n$ shall be uniform functions of the $n+r$ variables x, a, and shall be developable in positive integral powers of $x_i - x_i^0$, $a_k - a_k^0$, x_i^0 and a_k^0 being arbitrary points in the regions (x) and (a); while the Jacobian

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

vanishes for no possible combination of values within those regions. We notice further that now for any two distinct systems of values of the x_i, the equations $x'_i = f_i(x, a)$ yield two distinct values of the x'_i. Under these conditions the x'_i are confined to a certain region (x') and if we assign to the x'_i any system of values within this region, say x'^0_i, and to the a_k any system of values within (a), say a_k^0, then within the region of x'^0_i and a_k^0 the x_i are developable in ordinary power series and are uniform functions of the x'^0_i; in fact they are uniform throughout the entire region (x'). Now in order to make the substitution of the x'^ν from $x'_\nu = f_\nu(x, a)$ allowable in $x''^\nu = f_\nu(x', b)$, so as to obtain the equation $x''_\nu = f_\nu(x, c)$, we must clearly confine x and a to certain sub-regions $((x))$ and $((a))$ such that x'_ν lies within (x).

The differential equations satisfied by the group are so very important in the theory, since they lead to the infinitesimal transformations, that I shall venture to reproduce a part of the analysis which gives them. We differentiate the equations

$$f_i(x', \ldots x'_n, b_1, \ldots b_r) = f_i(x_1, \ldots x_n, c_1, \ldots c_r)$$

partially with respect to a_k, taking the x_i, c_k and a_k $[i = 1 \ldots n,\ k = 1 \ldots r]$ for independent variables. This is allowable since the only relations existing between the quantities are:

$$x'_i = f'_i(x, a);\ c_k = \varphi_k(a, b)\ [i = 1 \ldots n;\ k = 1 \ldots r].$$

* A group with r essential parameters is called r-branch.
† Gebiet.

This gives

$$\sum_1^n{}_j \frac{\partial f'_i}{\partial x'_j}\frac{\partial x'_j}{\partial a_k} + \sum_1^r{}_\mu \frac{\partial f'_i}{\partial b_\mu}\frac{\partial b_\mu}{\partial a_k} = 0 \; [k = 1 \ldots r, \; i = 1 \ldots n],$$

where f'_i stands for $f_i(x', b)$. The Jacobian by hypothesis does not vanish, hence we can solve for the $\frac{\partial x'_j}{\partial a_k}$ and obtain,

$$\frac{\partial x'_\nu}{\partial a_k} = \sum_1^r{}_j \Phi_{j\nu}(x', b) \frac{\partial b_j}{\partial a_k} \; [\nu = 1 \ldots n; \; k = 1 \ldots r].$$

It is important to notice that the $\Phi_{j\nu}$ are quite independent of the index k. Further, by using the relations

$$c_\mu = \varphi_\mu(a, b),$$

we may express the $\frac{\partial b_j}{\partial a_k}$ as functions of the a_k and b_k alone, and thus reach the expression

$$\frac{\partial x'_\nu}{\partial a_k} = \sum_1^r{}_j \Psi_{jk}(a, b) \; \Phi_{j\nu}(x', b) \; [\nu = 1 \ldots n; \; k = 1 \ldots r].$$

Whatever values the b_k may have, these equations are always identically satisfied by the substitution $x'_i = f_i(x, a)$; let us then assign fixed numerical values to the b_k so as to be rid of arbitrary constants. Writing $b_k = \omega_k$ and taking

$$\Phi_{j\nu}(x', \omega) = \xi_{j\nu}(x'_1, \ldots x'_n),$$

$$\Psi_{jk}(a, \omega) = \psi_{jk}(a'_1, \ldots a'_r),$$

we may write:

$$3) \qquad \frac{\partial x'_\nu}{\partial a_k} = \sum_1^r{}_j \psi_{jk}(a) = \xi_{j\nu}(x'_1, \ldots x'_n).$$

The solution of these equations clearly has the form:

$$\xi_{ki}(x'_1, \ldots x'_n) = \sum_1^r{}_j \alpha_{kj}(a) \frac{\partial x_i}{\partial a_j}.$$

Equations 3) are the fundamental differential equations of the group; the functions ξ which appear in them possess

the property that they satisfy no n relations $\sum_1^r{}_k e_k \xi_{ki} = 0$ $[i = 1 \ldots n]$, where the e_k are different from 0 and free from $x'_1, \ldots x'_n$. If then we form the r expressions $X_k(f) = \sum_1^n{}_i \xi_{ki}(x) \frac{\partial f}{\partial x_i}$, it is quite clear that the $X_k(f)$ satisfy no relation $\Sigma e_k X_k = 0$.

The beauty and success of Lie's analysis lies in his use of the expressions X_k,* which are the *infinitesimal transformations* of the group. We see from the above that from every r-branch † group we may obtain r infinitesimal transformations as soon as we have formed equations 3).

It is interesting to trace the course of thought which led Lie to designate the expressions as he did, though it is very clear and simple. In the case of a 1-branch group containing the identical transformation, equations 3) take the form

$$4) \qquad \frac{dx'_i}{dt} = \xi'_i(x'_1, \ldots x'_n),$$

t being the parameter, and the equations of the group may be obtained in the form

$$y'_1 = y_1, \ldots y'_{n-1} = y_{n-1}, \; y'_n = y_n + t$$

by integrating 4). If now this integration be performed with the initial condition that for $t = 0$, $x' = x$, Taylor's theorem gives

$$x'_i = x_i + \frac{t}{1} \xi_i(x) + \frac{t^2}{1.2} \sum_1^n{}_k \xi_k \frac{\partial \xi_i}{\partial x_k} + \ldots \; [i = 1 \ldots n].$$

Denoting $\sum_1^n{}_i \xi_i \frac{\partial f}{\partial x_i}$ by $X(f)$ and observing that $X(x_i) = \xi_i$, we may write:

$$x'_i = x_i + \frac{t}{1} X(x_i) + \frac{t^2}{1.2} X(X(x_i)) + \ldots$$

We pass from one transformation of the group to another by assigning different values to t, and if t be infinitesimal the value of x' will differ but little from that of x; in other words the transformation will be infinitesimal. We shall have in that case

* Of course Lie carefully calls attention to the previous use of similar symbols by Jacobi and Poisson. Abschnitt II., Kap. 7.

† Having r essential parameters.

$$x'_i = x_i + X(x_i)\,dt.$$

It is possible to express any function $f(x'_1, \ldots x'_n)$ in the same way

$$f(x'_1, \ldots x'_n) = f(x) + \frac{t}{1} X(f) + \ldots .$$

so that $X(f)\,dt$ really denotes the infinitesimal increment of the function when the parameter takes an increment dt. The connection between infinitesimal transformations and 1-branch groups is briefly this, that every infinitesimal transformation generates a definite 1-branch group, and every 1-branch group containing the identical transformation is generated by a definite infinitesimal transformation. There is no difficulty in showing* that when the group does not contain the identical transformation we can obtain all the transformations, at least within a certain region, by first effecting a particular, fixed, transformation of the group, and then a transformation of a group generated by infinitesimal transformations.

In a review article it is impossible to do more than point out with hasty gestures the landmarks which must guide the traveler who would explore the vast intellectual territory which Lie has opened up and minutely described; but it would be a wrong to the reader who has yet before him the pleasure of a first perusal of this book, to omit to mention the passages in which the author demonstrates the generation of the r-branch by the 1-branch groups. There are passages in mathematical works which possess all the facility, the grace, and harmony of construction which characterize purely literary productions of the highest order; but add to this charm the dramatic element which comes from the ordered movement of the demonstration, and we shall, perhaps, partially account for the reader's feeling as he peruses these pages.† It is sufficient here merely to state that the r-branch group containing the identical transformation and whose transformations can be arranged in pairs of "inverses," is generated by ∞^{r-1} 1-branch groups; and that other r-branch finite continuous groups can be obtained within a certain region by first performing a fixed transformation of the group and then a transformation of a certain 1-branch group whose infinitesimal transformation has the form $\sum_1^r{}_k \lambda_k X_k(f)$, where $X_1, \ldots X_k$ are the infinitesimal transformations given by equations 3).

It is not difficult to foresee that important relations must

* See p. 50. † See pp. 61–81.

exist between the group whose infinitesimal transformations are $X_1, \ldots X_r$ and the complete system of linear partial differential equations formed by equating to zero $X_1, \ldots X_r$ and such expressions of the form $(X_i X_j)$,* as are not linearly expressible in terms of $X_1, \ldots X_r$; in fact the solutions of the complete system are the invariants of the group. Let $\omega(x_1, \ldots x_n)$ be such a solution; then, as we have seen,

$$\omega(x'_1, \ldots x'_n) = \omega(x) + \frac{t}{1} \sum_1^r{}_k \lambda_k X_k(\omega)$$
$$+ \frac{t^2}{1.2} \sum_1^r{}_k \lambda_k X_k \left(\sum_1^r{}_k \lambda_k X_k(\omega) \right) + \ldots$$

But by hypothesis $X_k(\omega) = 0$, and, therefore, $X_i(X_k(\omega)) = 0$, and so on; hence, $\omega(x') = \omega(x)$, or, as it may be expressed, *the solutions of the complete system admit* all the transformations of the group.

When $X_1, \ldots X_r$ equated to zero form a complete system, it is well known that we have always

$$(X_j X_k) = \sum_1^r{}_\mu \psi_{\mu jk}(x_1, \ldots x_n) X_\mu,$$

but it is a remarkable fact that when $X_1, \ldots X_r$ are the infinitesimal transformations of an r-branch group the $\psi_{\mu jk}$ become absolute constants and we have

$$(X_j X_k) = \Sigma c_{\mu jk} X_\mu.$$

Certain relations evidently exist among the $c_{\mu jk}$, since $(X_j X_k) = -(X_k X_j)$ and since the Jacobian identity gives $[X_i(X_j X_k)] + [X_j(X_k X_i)] + [X_k(X_i X_j)] = 0$; it is therefore not true that any set of constants chosen at random will correspond to some group. The array of the constants $c_{\mu jk}$ is called by Lie the setting† of the group; when they are known it is possible to determine all the continuous sub-groups.

Let $Y_1, \ldots Y_m$ be the infinitesimal transformations of a sub-group of $X_1, \ldots X_k$. Then we must have $(Y_\mu Y_\nu) = l_{\mu\nu\pi} Y_\pi$, by definition of a sub-group. But also $(Y_\mu Y_\nu) = \sum_{\rho\sigma}^{1 \ldots \nu} h_{\mu\rho} h_{\nu\sigma} (X_\rho X_\sigma)$ since

$$Y_\mu = \sum_1^r{}_\rho h_{\mu\rho} X_\rho.$$

* This well-known symbol needs no explanation.
† Zusammensetzung.

Equating the two expressions for $(Y_\mu\ Y_\nu)$ it is found possible by purely algebraical operations to determine the various sets of $h_{\mu\rho}$ in terms of the $c_{\mu jk}$ and thus to determine all the subgroups of a given r-branch group.

We shall make no attempt here to follow the author in those investigations of special properties of groups which occupy a large part of the remainder of the book. The great and increasing importance of his subject would be sufficient of itself to call the attention of mathematical readers to Lie's volumes, but we hope, even at the risk of doing a work of supererogation, to supplement this sketch in the near future with an account of the second part of the Theory of Transformation Groups, which deals with the transformations of contact.

C. H. CHAPMAN.

MILWAUKEE, *November* 1, 1892.

THE THEORY OF FUNCTIONS OF A REAL VARIABLE.

ULISSE DINI, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse.* Deutsch bearbeitet von Dr. JACOB LÜROTH, Professor zu Freiburg i. B., und ADOLF SCHEPP, Premier Lieutenant A. D. zu Wiesbaden. Leipzig Teubner, 1892. 8vo, pp. xviii + 554.

THIS German version will be welcomed by many mathematicians who have been debarred from a study of Dini's classical treatise by unfamiliarity with the language. In the preface the translators state that they have not attempted to recast the materials, although they feel that some investigations might be transferred with advantage from the later to the earlier sections. Several paragraphs have been added, and the second half of the book has been divided into short chapters. The principal novelties are the introduction of Cantor's definition of the irrational number in place of Dedekind's, the proof that a continuum of points cannot be a mass of points of the first power, an account of Cantor's method for the condensation of singularities, and some additional theorems on integration. The usefulness of the book has been greatly increased by the insertion of numerous references to original sources, and by a list, at the end, of 45 authors whose works are referred to in the text. The translators have performed their difficult task with skill and judgment. We must call attention, however, to the numerous misprints which disfigure the text; for instance, § 8* is, as it stands, very obscure.

Such a work as this of Dini's raises the question as to how far refinements in analysis ought to be carried. Does science gain anything from researches on progressive and regressive differential quotients, on assemblages of points in close and loose order, on continuous functions with infinitely many maxima and minima in a region of finite extent? Or are these, and other curiosities, freely scattered through 500 pages mere mathematical toys of a somewhat higher order of excellence than the magic squares which used to please our ancestors? Eminent mathematicians are disagreed upon this point. Investigations which relate to such matters are, according to some writers, simply studies of disease, or contributions to the "morbid pathology of functions"; for they can have no bearing upon truly important problems of pure or applied mathematics. This view is, we believe, a mistaken one. It is worthy of note that some of the theorems which must have been startling in their novelty, when delivered in lectures at Pisa in 1871, have now become current mathematical coin; as, for example, those on limits, on continuous functions without differential quotients, on uniform convergence of series, on the integration and differentiation of series, etc. It is equally worthy of note that these theorems have either been discovered by, or occupied the attention of, many distinguished living mathematicians.

The first three chapters are preliminary to the discussion of the continuity and discontinuity of functions; they deal with irrational numbers, masses of points, limits, and oscillations. In chs. IV. to VI. functions are considered with regard to their continuity, or discontinuity; in ch. VII., with regard to the character of their differential quotients. Ch. VIII. treats of infinite series; ch. IX. of the condensation of singularities; ch. X. of continuous functions without differential quotients; ch. XI. of progressive and regressive differential quotients. In ch. XII. some important theorems are given, relating to the existence of differential quotients. As a specimen we may instance the following:

If a function $\varphi(x)$ is given, with finite and continuous differential quotients at the points of a given interval (a, b), to which x_0 belongs, then Taylor's series

$$\varphi(x_0) + (x-x_0)\varphi'(x_0) + \frac{(x-x_0)^2}{2!}\varphi''(x_0) + \ldots$$

is, generally, convergent throughout some interval $(x_0 - \rho, x_0 + \rho)$. This series represents a function $\psi(x)$ which is finite and continuous, together with all its differential quotients, at all points of the interval (with the possible exception of the terminal points). The functions $\varphi(x)$, $\psi(x)$ have

the same values and the same differential quotients at x_0, but may differ in these respects at all other points of the region common to (a, b) and $(x_0 - \rho, x_0 + \rho)$. It is instructive to contrast this theorem with Cauchy's theorem, that a monogenic function $\varphi(x)$ of a complex variable x, which is holomorphic in the domain of x_0, is equal to

$$\varphi(x_0) + (x - x_0)\,\varphi'(x_0) + \frac{(x - x_0)^2}{2!}\,\varphi''(x_0) + \ldots$$

throughout that domain.

Ch. XIII. to XX. are concerned with integration. Free use is made in them of the theory of masses of points. Any system of real numbers constitutes a mass P of real numbers; or, what amounts to the same thing, a mass of points on a straight line. This mass gives rise to derived masses $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, . . . The first derived mass consists of all the limiting points of the mass P. For instance, if

$$P = (1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}+\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{2}+\tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{2}+\tfrac{1}{5}, \ldots),$$

the limiting points are $0, \frac{1}{2}$. In this example there is no derived mass $P^{(2)}$; and the mass P is said to be of the first kind. If the number of derived masses is finite and $= k$, the original mass is said to be of the kth kind and of the first species; but if the number of derived masses is infinite, the mass is said to be of the second species. At each point of $P^{(1)}$ there is an infinite cluster of P-points, but the points of $P^{(1)}$ may be distinct from those of P. The points of $P^{(2)}$, $P^{(3)}$, . . . are all included among those of $P^{(1)}$. When P is identical with $P^{(1)}$ the mass is perfect. The mass which consists of all the rational numbers between 0 and 1 is imperfect. In this example the mass of points fills completely the interval $(0, 1)$; and the points are said to be in close order, since one point at least of the mass lies in any partial interval however small. If the discontinuities of a finite function $f(x)$ fill completely a certain interval, the function is not integrable for that interval. It may happen that a mass of infinitely many points is such that no segment of an assigned interval, such as $(0, 1)$, is filled completely. In this case the points are said to be in loose order. When the places of discontinuity of a finite function $f(x)$, for the assigned interval, are in loose order integration *may* be possible (Hankel fell into an error when he asserted that it *must* be possible); but if they are in close order, integration, in the sense usually accepted, is impossible. The editors mention in the preface that they omit the extension of the integral-concept to this case, because it would carry them too far into the calculus of the transfinite num-

bers. In ch. XIII. the passage to the new definition of a definite integral is skilfully effected by the use of the old definition

$$\int_\alpha^\beta \varphi'(x)\,dx = \varphi(\beta) - \varphi(\alpha).$$

It is shown that the integral of a finite function $f(x)$ in a finite interval (α, β), where $\beta > \alpha$, is the limit to which $\sum_1^n \delta_s f(x_{s-1} + \varepsilon_s \delta_s)$ tends, when d, the norm of the division, decreases below any assigned number. Here the x's are points interpolated between α, β; the sub-intervals δ are less than d, and the numbers ε lie between 0 and 1, but depend upon the numbers $x_1, x_2, \ldots, x_{n-1}$. The second step in the argument is the proof that the ε's can be chosen arbitrarily between 0 and 1; and next the oscillations $D_1, D_2, \ldots, D_n$ in the partial intervals are introduced. Finally, it is shown that the necessary and sufficient condition that the finite function $f(x)$ be integrable in the interval (α, β) is the following: to every arbitrarily small number σ there must correspond some system of partial intervals $\delta_1, \delta_2, \ldots, \delta_n$ such that the corresponding sum $\Sigma\, \delta_s D_s < \sigma$. Professor H. J. S. Smith has pointed out that the proof of this theorem is more difficult than might at first sight appear. It is not enough to show that

$$\delta_1 D_1 + \delta_2 D_2 + \ldots + \delta_n D_n$$

tends to a limit when all the δ's tend to 0, whatever be the law of division into sub-intervals. In the proof account must be taken of the fact that the greatest value of $\Sigma \delta_s D_s$ arises from a set of sub-intervals distinct from the set which makes $\Sigma \delta_s D_s$ a minimum. Many theorems are given which relate to the integration of finite functions whose discontinuity-points form a mass of the first species. For example, let $S = \sum_{s=1}^n \delta_s p_s$, where p_s is 1 or 0, according as a point of discontinuity falls in δ_s or not. If S has a lower limit 0, $f(x)$ is integrable in (α, β). There are still other divisions of masses. A mass is said to be numerable (*abzählbar*) when the points of the mass correspond, one-to-one, with the system of numbers 1, 2, 3, ...; such a mass is of the first power.

That this theory of masses is not without applications can be proved by a striking example. Jacobi's famous dictum with regard to functions of one variable "*functio tripliciter periodica non datur*" has led to many misconceptions; its accuracy has been disputed by Göpel and Casorati, and it has been shown that there exist infinitely many-valued functions of one variable which have more than two periods. Jacobi's

argument, as presented by Clebsch and Gordan, is briefly this: the absolute value of $m_1\omega_1 + m_2\omega_2 + \ldots + m_{2p}\omega_{2p}$ can be made less than any assigned number ε, however small, and therefore the Abelian integral of the first kind, with assigned upper and lower limits, can be made to take every value by a suitable choice of periods. This amounts to saying that the points $m_1\omega_1 + m_2\omega_2 + \ldots + m_{2p}\omega_{2p}$ form a continuum. But it can be shown that the points just mentioned form a numerable mass, whereas the points of a continuum cannot be made to correspond one-to-one with the system (1, 2, 3, . . .). Thus the conclusion drawn by Clebsch and Gordan is unwarranted; and the upper limit of an Abelian integral of the first kind may be regarded, legitimately, as an infinitely many-valued function of the integral itself. As Casorati has pointed out,* Fuchs makes the following assertion solely on the strength of Jacobi's theorem: "Es ist ein für die Grundlagen der Theorie der Differentialgleichungen wesentlicher Umstand, dass es auch unter den Differentialgleichungen jeder Ordnung und jeden Grades, welche nicht nur Transformationen von (α) sind, Classen solcher Art gibt, welche zwischen der unabhängigen und abhängigen Veränderlichen keine functionale Beziehung im gewöhnlichen Sinne des Wortes festsetzen, so lange jene Veränderlichen complexe Werthe annehmen dürfen" [here (α) refers to a differential equation $dx/du = \sqrt{R(x)}$]. This example shows that the theory of masses of points is not of purely theoretic interest. It would not be difficult to bring forward other examples. The theory of automorphic functions, recently developed by Poincaré and Klein, affords instances of analytic functions with natural boundaries, and attention in such cases is directed to the character of the mass of singular points. The importance of the analytic function of the complex variable is well known, and it is a matter of some consequence to have a method for the construction of analytic functions with linear barriers. In this connection the continuous function without a differential quotient becomes of use. M. Painlevé, in a memoir of marked originality,† has constructed such functions by means of the series

$$\sum_{n=1}^{\infty} \frac{\cos a_n\theta}{a_n}, \qquad \sum_{n=1}^{\infty} \frac{\sin a_n\theta}{a_n},$$

which, when $a_n = n^n$, or $n!$ are continuous functions of θ without differential quotients. The associated series

$$\sum_{n=1}^{\infty} \frac{z^{a_n}}{a_n},$$

* *Acta Mathematica*, vol. VIII., Les fonctions d'une variable à un nombre quelconque de périodes, p. 347.

† *Sur les lignes singulières des fonctions analytiques.* Paris, 1887.

represents an analytic function of z in a circle whose centre is 0 and radius 1; but the series is not continuable beyond the rim of the circle. Other examples of the use of Dini's theorems might be selected from Painlevé's memoir, from recent researches on Fourier's series, and from Dirichlet's problem in the theory of the potential.

May we, in conclusion, express the hope that some reader of the BULLETIN will follow the good example of Dr. Lüroth and Lieutenant Schepp, and translate this highly interesting work into English. After a student has become familiar with the ordinary processes of the infinitesimal calculus, it is highly desirable that he should have easy access to a special treatise in which attention is paid to fundamental principles rather than to details. Dini's treatise fulfils these requirements, and is at the same time flawless as regards rigour of proof and clearness of explanation.

J. HARKNESS.

BRYN MAWR, *December* 7, 1892.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, December 3, at half-past three o'clock, the vice-president, Professor Fine, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Professor Fabian Franklin, Johns Hopkins University; Dr. George W. Hill, West Nyack, N. Y. It was announced that the annual meeting would be held on Thursday afternoon, December 29, at four o'clock. A committee of three, consisting of Dr. Pierson, Dr. Stabler, and Mr. Maclay, was elected to report at the annual meeting nominations for the officers and other members of the council for the coming year. A paper by Professor J. W. Nicholson on "The expression of the nth power of any number in terms of the nth powers of other numbers, n being any positive integer," was read. Professor Fine called attention to a purely algebraic method, not involving the notion of continuity, for treating the theory of contact of algebraic curves.

THE recent circular issued by the committee on the proposed joint memorial at Göttingen to Gauss and Weber, contains the following remarks upon these two great investigators: "What both accomplished in the service of science is not the property of their pupils alone, but an inheritance of all mankind, which has already been, and which still in the future

will be of great importance to the advance of technology, commerce, and civilization.

"Gauss, unequalled in the universality of his genius among the scholars of the century, not only left monuments of his work in every region of pure mathematics, but enriched and extended the entire domain of mathematical astronomy and physics. His investigations are fundamental in theory and in observation. Whenever quantitative observations are made upon any natural phenomena, the observer, in order to derive his results, turns at once to rules furnished by Gauss. Of the methods discovered by him, however, those have been of the widest usefulness which gave a convenient and accurate quantitative definition of the physical agencies formerly supposed incapable of exact measurement, expressing these so called imponderables in terms of the absolute units of length, time, and mass.

"What Gauss did in this respect for magnetism, was done for the intensity of the electric current, electromotive force and resistance by Weber, who was chosen as a fellow worker by the older investigator on account of his early work in acoustics. When the latter had shown how to measure these quantities in absolute units, independent of the conditions of observation, he had afforded not only to the progress of science assistance of extraordinary value, but also to electrotechnics an instrument indispensable to its practice, the excellence of which is demonstrated by its universal acceptance, and to which is due, in no slight degree, the gigantic stride of technology witnessed during the past decade.

"The object of these lines prevents us from entering more fully into the other results which we owe to the common labors of these great men. We may recall only their fruitful investigations of the laws of terrestrial magnetism, from which, in a way, a new branch of physics has grown; their attempts to include within a single law the phenomena of statical electricity, dynamical electricity, and induction, attempts which, whether successful or not, mark an epoch in the historical development of electrical science; and, finally, the most popular result of their collaboration, the erection of the first practicable long distance telegraph. Of the other works of Weber we may mention here only his determination, jointly with R. Kohlrausch, of the ratio of the electrostatic and electromagnetic units of current, which gave a direct impulse to the most recent development of the theory of electricity and the closely related electrical theory of light.

"The birthplace of Gauss, since 1877, has possessed a monument erected in his honor by scholars all over the world, but Göttingen, where he, like Weber, displayed by far the greatest part of his activity, still lacks such a memorial."

The circular states, in closing, that contributions to the memorial will be received up to April 1, 1893, and that they should be addressed to the banking-house of Siegfried Benfey, in Göttingen. The secretary of the NEW YORK MATHEMATICAL SOCIETY will take pleasure in forwarding any contributions which may be sent to him.

AT Oxford University, during the Michaelmas Term just ended, the following courses of lectures on mathematical subjects were delivered: Professor J. J. Sylvester, Invariants and covariants of systems of conics; Professor Bartholomew Price, Theory of fluids and of fluid motion; Professor C. Pritchard, Theory and computation of eclipses, and their phenomena; Mr. A. L. Pedder, Elementary solid geometry; Mr. E. B. Elliott, Higher integral calculus; Mr. J. E. Campbell, Differential equations; Mr. H. T. Gerrans, Rigid dynamics; Mr. C. E. Haselfoot, Algebra; Mr. J. W. Russell, Series and continued fractions; Mr. C. Leudesdorf, Projective geometry; Mr. C. H. Sampson, Analytical geometry; Mr. W. Esson, Theory of plane curves; Mr. E. H. Hayes, Elementary mechanics.

T. S. F.

NEW PUBLICATIONS.

COMPILED BY B. WESTERMANN & CO., NEW YORK.

I. HIGHER MATHEMATICS.

AUTONNE (L.). Sur la théorie des équations différentielles du premier ordre et du premier degré. Paris, Masson, 1892. 8vo. Fr. 9.00

BACHMANN (P.). Die Elemente der Zahlentheorie. Leipzig, Teubner, 1892. 8vo. 12 and 264 pp. Mk. 6.40

BERGBOHM (J.). Entwurf einer neuen Integralrechnung auf Grund der Potenzial-, Logarithmal- und Numeralrechnung. Die rationalen algebraischen und die goniometrischen Integrale. Leipzig, Teubner, 1892. 8vo. 6 and 66 pp. Mk. 1.00

CHAPMAN (C. H.). An elementary course in the theory of equations, for the junior year in colleges. Treating of determinants, algebraic equations, and the computation of the real roots of numerical equations. New York, Wiley, 1892. 12mo. Cloth. $1.50

HAAS (A.). Lehrbuch der Differentialrechnung, nach System Kleyer bearbeitet. Theil II : Die vollständige Differentiation entwickelter und nichtentwickelter Functionen von einer und von mehreren reellen Veränderlichen ; Reihenentwicklungen ; unbestimmte Formen ; Maxima und Minima. Stuttgart, Maier, 1892. 8vo. fig. Mk. 8.00

JOHNSON (W. W.). The theory of errors and the method of least squares. New York, Wiley, 1892. 12mo. 180 pp. Cloth. $1.50

KÜHNE (H.). Beiträge zur Lehre von der n-fachen Mannigfaltigkeit. Greifswald, 1892. 8vo. 55 pp. Mk. 1.50

MILLER (G. A.). An introduction to the study of determinants, with examples and applications. New York, Van Nostrand, 1892. 16mo. 8 and 110 pp. Boards. (Van Nostrand's science series, No. 105.) $0.50

MONTESANO (D.). La rappresentazione su di un piano delle congruenze di rette di secondo ordine dotate di linea singolare. Potenza, Spera, 1892. 8vo. 25 pp.

———. Su di un complesso di rette di terzo grado. Potenza, Spera, 1892. 8vo. 32 pp.

MOTTI (G.). Risoluzione della quadratura del circolo e quadratrice di Dinostrato con modificazione ai valori delle formole del pendolo contenenti il π d'Archimede ; quadratura dell' ellisse. Pavia, Marelli, 1892. 8vo. 64 pp. 1 plate.

RUDIO (F.). Vier Abhandlungen über die Kreismessung von Archimedes, Huygens, Lambert und Legendre. Deutsch herausgegeben und mit einer Übersicht über die Geschichte des Problems von der Quadratur des Zirkels, von den ältesten Zeiten bis auf unsere Tage, versehen. Leipzig, Teubner, 1892. 8vo. 166 pp. fig. Mk. 4.00

THÉON DE SMYRNE. Exposition des connaissances mathématiques utiles pour la lecture de Platon, traduite pour la première fois du grec en français, par J. Dupuis. 8vo. Paris, Hachette, 1892. Fr. 7.50

WEIERSTRASS (K.). Formeln und Lehrsätze zum Gebrauch der elliptischen Functionen. Nach seinen Vorlesungen und Aufzeichnungen bearbeitet und herausgegeben von H. A. Schwarz. Berlin, Springer. 1892. 4to. 96 pp. Mk. 10.00

II. ELEMENTARY MATHEMATICS.

BAILEY (M. A.). American mental arithmetic. New York, American Book Company, [1892]. 16mo. 160 pp. Cloth. $0.35

BOWSER (E. A.). A treatise on plane and spherical trigonometry, and its applications to astronomy and geodesy, with numerous examples. Boston, Heath, 1892. 8vo. 14 and 368 pp. $1.50

———. Elements of plane and spherical trigonometry, with numerous examples. Boston, Heath, 1892. 8vo. 6 and 172 pp. $0.90

———. The elements of plane geometry, with numerous exercises. 3d ed. Boston, Heath, 1892. 8vo. 6 and 250 pp. $0.75

BRENT (D.). Euclid, books 1 and 2. With notes and exercises. London, 1892. 8vo. Cloth. 2s.

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THE THEORY OF SUBSTITUTIONS.

The Theory of Substitutions and its Applications to Algebra. By Dr. EUGEN NETTO, Professor of Mathematics in the University of Giessen. Revised by the Author and translated with his permission by F. N. COLE, Ph.D., Assistant Professor of Mathematics in the University of Michigan. Ann Arbor, Mich., The Register Publishing Company, 1892. 8vo, pp. xii. + 301.

NETTO'S "Substitutionentheorie und ihre Anwendungen auf die Algebra" appeared for the first time in 1882; it was followed, in 1885, by an Italian edition, and now we have the pleasure of welcoming an English edition, revised by the author and translated into English by Dr. Cole.

The mathematical public at large, and the English-speaking part of it in particular, are greatly indebted to Dr. Cole for his careful and expert translation. Mastering the subject as well as both languages in full extent, Dr. Cole has transformed the sometimes rather tough material into clear and fluent English. We are especially obliged to him for the fortunate choice of many technical terms, alien so far to the English mathematical language.

We are equally indebted to the author for the numerous valuable additions by which this new edition has been enlarged and improved.

The great merit of Netto's book consists in the skilful and highly pedagogical presentation of the theory of substitutions, given in the first part of the book. The reader is gradually led from the most elementary considerations on symmetric and alternating functions to the general theory of unsymmetric functions of n independent elements, out of which the theory of substitutions is step by step evolved, the unsymmetric functions serving all the while as a concrete substratum for the abstract conclusions of the theory of substitutions. By this means an easy and attractive entrance into the theory of substitutions is gained, accessible even to the beginner, and it may fairly be said that Netto's book has largely contributed to spread the knowledge of this important branch of mathematics.

While thus fully acknowledging the high and lasting merits of the first part of Netto's book, we cannot withhold our opinion that the author has not been equally successful in his attempt to simplify Galois' theory of the algebraic solution of equations, which forms the principal subject of the second part. The great difficulties which are contained in Galois' theory, are, it seems to us, not sufficiently considered, and the comparative simplicity of the deductions is only obtained at the cost of rigor.

The following review* will be divided into four sections :

I. The first part of Netto's book.

II. Excursus on the principal difficulties of Galois' theory.

III. Analysis of Netto's exposition of the theory of the group of an equation.

IV. The remaining chapters of the second part.

I.

The first part is devoted to the theory of substitutions and of integral functions. It opens in chapter I., with an exposition of the principal properties of *symmetric functions* of n independent quantities $x_1, x_2, \ldots x_n$. A symmetric function remains unchanged in form, and consequently also in value, when the x's are permuted in any way, and is therefore, at the same time, a *single-valued* function ; conversely every single-valued function of n independent elements is symmetric in these elements. The simplest symmetric functions are the elementary symmetric functions :

$$\begin{aligned} c_1 &= x_1 + x_2 + \ldots + x_n \\ c_2 &= x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \\ &\vdots \\ c_n &= x_1 x_2 \ldots x_n. \end{aligned}$$

Every integral symmetric function of $x_1, x_2, \ldots x_n$ can be expressed as an integral function of the elementary symmetric functions $c_1, c_2, \ldots c_n$.

If a function is not symmetric, it will be changed in form, and consequently, if the x's are independent, also in value, by some of the possible interchanges of the x's. We shall have two-valued, three-valued, m-valued functions, according to the number of different values they take when the x's are permuted in all possible ways. The simplest unsymmetric functions are the *two-valued functions*. The square root of the discriminant

$$\sqrt{\Delta} = \prod_{\lambda, \mu} (x_\lambda - x_\mu)$$

offers a first example of a two-valued function ; it is, moreover, an *alternating* function, because its two values differ only in sign. Every integral alternating function is the product of $\sqrt{\Delta}$ into an integral symmetric function. Every integral

* Reviews of the first edition will be found in : *Fortschritte der Mathematik*, 1882, p. 90. *Zeitschrift für Mathematik*, 1883, part II., p. 181. *Bulletin des Sciences Mathématiques*, 1883, p. 57.

two-valued function is of the form $S_1 + S_2\sqrt{\Delta}$, (where S_1 and S_2 are integral symmetric functions) and satisfies an equation of the second degree whose coefficients are integral functions of $c_1, c_2, \ldots c_n$.

If we pass to functions of a greater number of values, we soon perceive that the knowledge of the number of its values is not sufficient to characterize an unsymmetric function; we must go a step further and turn our attention from the functions to the interchanges of the x's, and in doing so, we enter the peculiar domain of the *theory of substitutions.*

After a few necessary explanations concerning notation, terminology, and some elementary properties of substitutions, the most important conception of the whole theory is introduced, that of a *group of substitutions*, a group being defined as a system of substitutions which reproduces itself by multiplication of its individual members. Those substitutions which leave a function $\varphi(x_1, x_2, \ldots x_n)$ unchanged, constitute a group called the group of φ. And *vice versa*: for every group of substitutions there are functions which are unchanged by all the substitutions of the group and by no others. This theorem contains the basis of a classification of functions of n elements: each class ("family"*) contains all the functions which belong to the same group.

Here arises the problem: To determine all the possible groups of substitutions of n elements.

The general solution † of this problem, however, presents difficulties as yet insuperable. Still, a great number of theorems concerning the construction of groups are known, and a first series of them are given in the remaining sections of chap. II. They refer to the symmetric group, the alternating group, the cyclic groups, and others of a more special nature.

Chapter III. is devoted to the different values of a multiple-valued function. Here we have first the theorem: The order r of a substitution-group G is always a divisor of $n!$; and every function φ which belongs to G, takes

$$\rho = \frac{n!}{r}$$

different values ("conjugate values") when operated upon by all the $n!$ substitutions. These results concerning the relation of the symmetric group and one of its subgroups can

* The term is only introduced in chap. V.

† That is for an indeterminate n. For a numerically given value of n, the problem can always be solved by a finite number of trials: the solution has been pushed as far as $n = 9$: compare several papers by ASKWITH and CAYLEY, *Quarterly Journal of Mathematics*, 1890, 1891, 1892.

be extended to the relation between any group and one of its subgroups.

Every one of the ρ conjugate values of a ρ-valued function belongs itself to a certain group; thus we obtain, corresponding to the ρ values, ρ *conjugate groups*. They are found to be "similar," and can be derived from the group G by the process of "transformation."

If it happens that these ρ groups coincide, or in other words, that the ρ conjugate values all belong to the same group, then G is called—to anticipate a term introduced only in the following chapter—a *self-conjugate subgroup* of the symmetric group. The alternating group is always a self-conjugate subgroup of the symmetric group, and (excepting the case $n = 4$) it is, beside the trivial group 1, the only self-conjugate subgroup of the symmetric group. On the other hand, the ρ conjugate values of our integral function φ are the roots of an equation of degree ρ whose coefficients are integral functions of the elementary symmetric functions $c_1, c_2, \ldots c_n$. And if now we ask under what circumstances this equation becomes binomial, it is easily shown that the group of the function φ must be a self-conjugate subgroup of the symmetric group. Combining this with the above result, we are led to the theorem:

The only unsymmetric functions of which a power can be symmetric, are the alternating functions.

Three different proofs of this important theorem are given; one of them is based on the properties of the *discriminant* of φ:

$$\Delta_\phi = \prod_{\alpha, \beta} (\varphi_\alpha - \varphi_\beta)^2$$

Along with these developments, we find, in chap. III. other investigations of a different nature. The problem of the *construction of groups* is again taken up, and a new and simplified proof of Cauchy-Sylow's theorem is given. As a new addition, a very interesting theorem due to Netto deserves special notice:

Suppose a group H of order h contains in one of its substitutions a cycle of order k, say $\omega_1 = (x_1, x_2, \ldots x_k)$. If h' is the order of that subgroup of H whose substitutions do not affect the k letters $x_1, x_2, \ldots x_k$, then h is divisible by the product $h'k$. And if now we transform the cycle ω_1 with respect to all the substitutions of H, we obtain $h/h'k$ distinct "conjugate cycles." Every one of these $h/h'k$ cycles occurs h' times in the group H; hence, the cycle ω_1 and its conjugates with respect to H occur h/k times in the group H. The number of letters in all these cycles is therefore equal to the order h of the group H. The special case $k = 1$ leads to a theorem previously given by Frobenius.

The fourth chapter treats of four properties of groups which are, each in its peculiar way, of fundamental importance for the algebraic solution of equations: transitivity, primitivity, composition and isomorphism.

a) A group is called *transitive* if its substitutions permit any selected element to be replaced by every other element; otherwise intransitive. Extending this definition to a set of k elements, we obtain the idea of a k-fold transitive group. The alternating group is $(n-2)$-fold transitive. The principal problem concerning transitive groups is the determination of an upper limit for the degree of transitivity for a group of degree n, which is neither the symmetric nor the alternating group. A series of special theorems due to Mathieu, Jordan and Netto, are given.* A new term is introduced, the *class of a substitution:* a substitution is of the kth class if it affects exactly k letters.

b) Closely connected with the idea of transitivity is that of *primitivity* and *non-primitivity*. A simply transitive group is called non-primitive when its elements can be divided into systems, each including the same number, such that every substitution of the group replaces all the elements of any system either by the elements of the same system or by those of another system. This section is entirely rearranged in the new edition and numerous new researches are added. We call particular attention to §§ 66, 67, where the relation between a non-primitive group and the corresponding group of substitutions of the systems is explained.

c) The fundamental idea of the *decomposition* of a group, due to Galois, has been touched a first time in the preceding chapter. Here the entire theory is developed.

A subgroup H of a group G is called *self-conjugate,* if it is commutative with all the substitutions of G, in symbols, if

$$t^{-1}Ht = H,$$

for every substitution t of G. A group which contains a self-conjugate subgroup, different from 1, is called a *compound group,* otherwise a *simple* group. If G contains no other self-conjugate subgroup K which includes H, then H is a *maximal* self-conjugate subgroup. The *series of composition* of a group G is a series of groups, beginning with G, ending with 1, such that each group is a maximal self-conjugate subgroup of the group immediately preceding it. The quotients of the orders of two consecutive groups are called the *factors* of composition of G. A compound group may admit of dif-

* For a further development of this theory we refer to two papers by Bochert, *Mathematische Annalen*, vols. 29 and 33.

ferent series of composition; in this case the factors of composition in the different series are identical, apart from their order. The series of composition of the symmetric group of n elements consists, if $n \gtrless 4$, of the alternating group and the group 1; for $n \gtrless 4$ the alternating group is simple.

d) *Isomorphism* is, in the simplest case, a one-to-one relation between the substitutions of two groups of the same order of such a nature that to the product of any two substitutions of the one corresponds the product of the two corresponding substitutions of the other group. The definition can be extended so as to apply also to a one-to-q-correspondence, and even to a p-to-q-correspondence: p-q-fold isomorphism. Isomorphic groups have a certain class of properties in common: for instance, in two simply isomorphic groups corresponding substitutions are of the same order; to a (self-conjugate) subgroup in the one corresponds a (self-conjugate) subgroup in the other, and so on.*

A very welcome addition is the introduction of the *quotient-group*, which plays such an important part in numerous recent researches. The easiest, though not best, way of defining it is perhaps the following: Let G be a group of order mk, H a self-conjugate subgroup of G of order m, $\varphi(x_1, x_2, \ldots x_n)$ a function belonging to H. The function φ, on being operated upon by all the substitutions of G, takes k values, $\varphi_1 = \varphi$, $\varphi_2, \ldots \varphi_k$. If now we apply to these k functions simultaneously all the substitutions of G, the φ's undergo a transitive substitution-group T of order k, which is 1-m-fold isomorphic with G. This group T is called, according to Hölder, the quotient of G and H, and is denoted by

$$T = G : H.$$

Netto adopts Hölder's definition, which is more abstract and independent of the function φ.† Likewise Hölder's *factor-groups* (the quotients of two consecutive groups in a series of composition) are mentioned.

In chapter V. Netto returns to the discussion of rational functions of n independent quantities, continuing and generalizing the investigations of chapter III.

The fundamental problem is: Given (a) a group G and one of its subgroups, H; (b) a rational function φ belonging to G and a rational function ψ belonging to H. What algebraic relation exists between φ and ψ? The answer is given in two theorems due to Lagrange:

---

* See below the remarks on groups of operations.

† Another very elegant definition is given by Dyck, *Math. Annalen*, vol. 20, pp. 11–15.

I.) φ is expressible as a rational function of ψ and of the elementary symmetric functions $c_1, c_2, \ldots c_n$ of $x_1, x_2, \ldots x_n$. This rational function is of the form :

$$\varphi = \frac{g(\psi\,;\, c_1, c_2 \ldots c_n)}{\Delta_\psi}$$

where g is an integral function of $\psi, c_1, c_2, \ldots c_n$, whereas Δ_ψ denotes the discriminant of ψ.

Two special cases of this theorem are particularly important :

1) Two functions belonging to the same group can be rationally expressed one in terms of the other. Hence the totality of all rational functions belonging to the same group form indeed a "family" (*Gattung*) in the sense in which Kronecker uses the word.

2) Every rational function of n independent elements can be rationally expressed in terms of every $n!$-valued function ; for instance in terms of

$$\xi = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n,$$

where the α's are arbitrary parameters.

II.) If the order of G is m times that of H, ψ satisfies an algebraic equation of degree m whose coefficients are rational functions of φ and $c_1, c_2, \ldots c_n$.

Here again the important question arises : Under what circumstances can this equation become binomial ? The answer is : H must be a self-conjugate subgroup of G ; conversely, if H is a self-conjugate subgroup of G, and if moreover m is a prime number, then there exist always, in the family of ψ, functions for which the equation becomes binomial. These results furnish a complete insight into the mechanism of the solution of the general equations of the second, third, and fourth degrees. Anticipating developments of chap. x., the general principle of their solution may be exhibited as follows : Let

$$G - H - I - \ldots . - 1$$

be the series of composition of a group G ; and suppose that all the factors of composition of G are prime numbers. On the strength of the last proposition, it is then possible to determine a series of functions $\varphi, \psi, \chi \ldots \xi$, belonging to the groups $G, H, I, \ldots 1$ respectively, and such that each function can be obtained from the function immediately preceding it by solving a binomial equation.

Now for $n = 2, 3, 4$, all the factors of composition of the symmetric group are prime numbers ; we may therefore apply

this principle to the symmetric groups of 2, 3, 4 elements and thus obtain the solution of the general equations of the degrees 2, 3, 4 by means of a chain of binomial equations. For the details, see §§148 to 150.

The remaining three chapters of the first part are devoted to special investigations concerning the existence or construction of substitution-groups between a given number of elements. We must confine ourselves to a short enumeration of the principal problems.

We have seen that the *number ρ of values of a multiple-valued function* of n elements is always a divisor of $n!$ This is a first strong restriction upon the possible values of ρ. Further restrictions are given in a series of theorems due to Bertrand, Cauchy, Serret, and Jordan.*

Every substitution-group is simply isomorphic with a *regular group*, that is, a transitive group whose order equals the number of elements; hence the importance of regular groups. Netto determines all regular groups whose order is either a prime number or the product of two prime numbers.

The circular substitution $(x_0, x_1, \ldots x_{p-1})$ may be written in an *abbreviated form*

$$\begin{pmatrix} x_z \\ x_{z+1} \end{pmatrix}, \ (z = 0, 1, 2, \ldots p-1),$$

or still shorter

$$| \, z \quad z+1 \, | \ (\text{mod } p),$$

if we agree to consider two indices as identical when they are congruent (mod p). In a similar way, the more general symbol

$$| \, z \quad \beta z + \alpha \, | \ (\text{mod } p)$$

in which α, β are two integers and β not $\equiv 0$ (mod p), represents a substitution if p is a prime number. The aggregate of all substitutions which are obtained by giving α, β all admissible values, form a group of order $p(p-1)$, called the *metacyclic group*. These results may be generalized in several directions: Either we may pass to *fractional* linear substitutions

$$\left| \, z \quad \frac{\alpha z + \beta}{\gamma z + \delta} \right| (\text{mod } p), \ (z = 0, 1, \ldots p-1, \infty);$$

they form a group of order $(p+1)\,p\,(p-1)$ which is of great importance in the theory of modular equations.

* For a further development of this difficult part of the theory we refer to two papers by BOCHERT, *Math. Annalen*, vols. 33 and 40.

Or we may introduce, instead of a single index z, a system of indices $z_1, z_2, \ldots z_k$; this leads to the theory of "*arithmetic*" and "*geometric*" substitutions, which plays an important part in the theory of solvable equations.

Or we may pass to the general theory of the *analytical representation* of substitutions (Hermite).

All these problems are studied in detail.

A particular interest attaches to the last subject which we have to mention, Kronecker's investigations on groups of *commutative substitutions*, not only on account of their importance for the theory of Abelian equations, but because, transgressing the limits of the theory of substitutions, they point to a more general field of research, the theory of *groups of operations* in general. With this one exception, Netto, true to his programme clearly expressed in the preface, strictly confines himself to groups of substitutions proper. Yet the general theory of groups throws such a new light backwards on the theory of substitution-groups out of which it has developed, that a few additional remarks on this important extension of the theory of substitutions may not be out of place.

It seems that Cayley * was the first to conceive the idea of extending the conception of a group, originally restricted to substitutions, to any operations which admit of repetition and combination. From this standpoint, two groups are considered as identical if the laws of combination of their operations are the same for both, no matter what the particular nature of the operations may be, whether they are substitutions, or rotations, or quaternions, or linear transformations, etc. Using the modern terminology, we may express the same idea in another form: Simply isomorphic groups are considered as identical. A systematic exposition of such a general theory of groups has first been given by Dyck.†

Now all possible properties of substitution-groups may be divided into two classes:

I.) Properties which are independent of the peculiar nature of the operations and are therefore properties of groups of operations in general. Such are the theorems on the relations between a group and its subgroups, on cyclic groups, Cauchy-Sylow's theorem,‡ and those on self-conjugate subgroups, on the construction of groups of a given order, on isomorphism, etc. Their common characteristic is that they are not destroyed by a transition to any simply isomorphic group.

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* On the theory of groups, *Phil. Mag.*, 4th series, vols. 7 and 18, and *American Journal of Mathematics*, vols. 1 and 11.

† *Math. Annalen*, vols. 20 and 22; besides, we refer to KLEIN, "Vorlesungen über das Ikosaeder," pp. 5–8, and HÖLDER, *Math. Annalen*, vol. 34, pp. 28–39.

‡ See FROBENIUS's proof, *Journal für Mathematik*, vol. 100, p. 179.

II.) Properties which depend on the peculiar nature of the operations, as the theorems on the symmetric and alternating groups, on transitivity and primitivity, on the construction of groups of a given degree, on the analytical representation of substitutions, etc.

An analogon, familiar to all, may serve to illustrate this classification: In the theory of algebraic curves, we distinguish between those properties of a curve which are not destroyed by projection and those which are destroyed. To the transition from one curve to another curve, projective with the first, corresponds in our case the transition from one group to another group, simply isomorphic with the first.

II.

Throughout the first part of Netto's book, the elements $x_1, x_2, \ldots x_n$ are regarded as entirely independent quantities.* We may therefore consider them as the roots of the *general equation* of the nth degree

$$(x - x_1)(x - x_2) \ldots (x - x_n) \equiv x^n - c_1 x^{n-1} + c_2 x^{n-2} \ldots \pm c_n = 0,$$

whose coefficients are themselves independent quantities. And all our theorems on rational functions of n independent quantities may be interpreted as theorems on rational functions of the roots of the general equation of the nth degree.

At first sight nothing seems simpler than the transition from the general equation to any *special equation*, that is, an equation whose coefficients are no longer independent quantities, but have either numerically given values or else are functions of one or more independent parameters. And yet, in this transition there are difficulties hidden which, it seems to us, can scarcely be too much emphasized. Before entering into a discussion of the second part of Netto's book, it will therefore be well to premise some remarks concerning the transition from general to special equations.

1) *First difficulty: domain of rationality.*

The peculiar difficulty of the theory of special equations faces us right at the outset when we try to define what we mean by saying an equation is *algebraically solvable.* The problem may be considered from two entirely different standpoints; it may either be regarded as a problem of the theory of functions or of pure algebra. Accordingly, we obtain two different definitions. We may say

* With the exception of a few special investigations, such as §§ 32, 111.

either (*theory of functions*):	or (*algebra*):
An equation is algebraically solvable, if its roots can be derived from the coefficients	
and a number of constant quantities	*and the numerical unity* 1*

by means of the five elementary algebraic operations, viz.: addition, subtraction, multiplication, division, extraction of roots of prime index, applied a finite number of times.

The difference between the two definitions is precisely the characteristic difference between theory of functions and algebra: in the theory of functions the fundamental distinction refers to variable quantities and constant quantities; our whole attention is directed toward the variables, and the arithmetical nature of the constants is left out of consideration. In algebra, on the contrary, the principal distinction refers to rational quantities and irrational quantities, and the chief interest concentrates upon the arithmetical nature of the constants.

The theory-of-functions-definition is applicable only to equations whose coefficients contain at least one variable parameter; if applied to numerical equations it would, combined with Gauss's theorem on the existence of the roots of an algebraic equation, lead to the result that every numerical equation is algebraically solvable.

The algebraic definition, with which we are here exclusively concerned, is applicable to all algebraic equations, whether the coefficients contain variable parameters or not. It admits, however, of a generalization which is essential for our further developments: The coefficients may themselves be derived from other quantities, regarded as known, by means of rational operations. Denoting the latter quantities by

$$\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''' \ldots,$$

an equation whose coefficients are expressible as rational functions *with integral coefficients* of $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''' \ldots$, will be said to be algebraically solvable, if its roots can be derived from the quantities $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''' \ldots$, by means of the above-named five elementary algebraic operations, applied a finite number of times. We expressly include in this definition the case in which some of the $\mathfrak{R}$'s do not explicitly appear in the coefficients; thus the coefficients of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

* The words "and the numerical unity 1" may be omitted since $1 = \frac{c}{c}$.

may be said to be rationally expressible in terms of the quantity $\mathfrak{R}' = \sqrt{5}$. It is therefore partly a matter of free choice from what quantities we regard the coefficients as being rationally derived. And the question whether a given equation is algebraically solvable admits of an answer only with respect to a definite choice of the $\mathfrak{R}$'s, or, to use Kronecker's terminology, it depends on the *domain of rationality* in which we are operating.

The domain of rationality ($\mathfrak{R}'$, $\mathfrak{R}''$, $\mathfrak{R}'''$ . . .) consists of all quantities which are expressible as rational functions with integral coefficients of $\mathfrak{R}'$, $\mathfrak{R}''$, $\mathfrak{R}'''$ . . . . The simplest domain of rationality consists of all rational numbers, and is characterized by a single quantity $\mathfrak{R}$, viz.: $\mathfrak{R}' = 1$.*

The conception of domain of rationality is of the most fundamental importance for Galois' theory; there is not a proposition in the whole theory in which it is not implicitly contained. And it should always be borne in mind that terms like "algebraically solvable," "irreducible," "group of an equation," "Abelian equation," and the like, have a meaning only if referred to some definite domain of rationality.

Again, if we say, a quantity, ξ, is a rational function † of the roots $x_1, x_2 \ldots x_n$ of our equation, say

$$\xi = \varphi\,(x_1, x_2 \ldots x_n),$$

we always mean a rational function whose coefficients belong to our domain of rationality. Without some restriction concerning the nature of the coefficients of $\varphi\,(x_1, x_2 \ldots x_n)$, the above statement would be meaningless, at least in the case of a numerical equation; and the idea of a rational function with any constant coefficients, as used in the theory of functions, should carefully be kept out of all purely algebraic investigations.

2) *Second difficulty: formal and numerical invariance.*

As long as the x's are undetermined quantities, two rational functions of $x_1, x_2, \ldots x_n$ are considered as equal only if they are identical, that is, equal for all sets of values of $x_1, x_2 \ldots x_n$. But if the x's are the roots of a special equation, it may hap-

* Compare ABEL, Œuvres, II., pp. 219, 220; GALOIS, Œuvres, p. 84 (*Journal de Mathématiques*, vol. 11, 1846); KRONECKER, *Journal für Mathematik*, vol. 92, pp. 3–10.

† There seems, however, to be something unsatisfactory in the use of the word rational function in this connection, particularly in the case of a numerical equation. "Function" always implies the idea of a variable quantity dependent on some other variable quantities, whereas in the present case all quantities involved are constant. We should prefer to say: ξ is rationally expressible in terms of $x_1, x_2, \ldots x_n$, or shorter ξ is rational in $x_1, x_2 \ldots x_n$. See HÖLDER, *Math. Ann.*, vol. 34, p. 41.

pen that two rational functions of $x_1, x_2, \ldots x_n$, though different in form are equal in numerical value.

Example: The roots of the equation

$$x^4 + 1 = 0$$

are, if we put $e^{\frac{\pi i}{4}} = \varepsilon$,

$$x_1 = \varepsilon,\ x_2 = i\varepsilon,\ x_3 = -\varepsilon,\ x_4 = -i\varepsilon.$$

The two functions

$$\varphi = x_1^2 \quad \text{and} \quad \psi = x_2 x_4,$$

though different in form, are numerically equal, viz. $= i$.

In particular, it may happen, that two conjugate values of a rational function become equal, in which case we have to distinguish between formal and numerical invariance. In the above example the function $\varphi = x_1^2$ remains *formally* unchanged by the substitutions: 1; (234); (243); (34); (42); (23), whereas it remains *numerically* unchanged not only by these substitutions, but besides by the substitutions: (13); (42) (13); (413); (4213); (4132); (213), which replace x_1 by x_3 (notice $x_1^2 = x_3^2 = i$).

a) *Formal invariance:* Those substitutions which leave a rational function $\varphi(x_1, x_2 \ldots x_n)$ formally unchanged form, of course, a group, as before. It must, however, be remembered that one and the same quantity, rational in the roots, may admit of different expressions in terms of the roots, in which case it may at the same time belong to several groups, according to its different expressions in terms of the roots. It seems therefore advisable to avoid in this connection the term "group of a rational function," or "group of a quantity rational in the roots," and only to speak of the group of a given "expression of a rational function."* Thus we would say, in the above example, the "expression" x_1^2 belongs to the group [1; (234); (243); (34); (42); 23], the "expression" x_2x_4 to the group [1; (24); (13); (13) (24)], whereas the "quantity" $i = x_1^2 = x_2x_4$ may be said to belong to both groups at the same time.

b) *Numerical invariance:* Those substitutions which leave a rational function $\varphi(x_1, x_2 \ldots x_n)$ *numerically* unchanged *do not in general form a group.*† In the above example the 12 substitutions which leave x_1^2 numerically unchanged do not form a group.

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* This is, however, only intended as a temporary terminology; the proper way of modifying the definition of the group of a rational function of the roots will be pointed out later: see p. 100, footnote.

† See Hölder, *Math. Annalen*, vol. 34, p. 41.

3) *Third difficulty: Lagrange's theorem.*

If some of the conjugate values of a rational function $\psi(x_1, x_2 \ldots x_n)$ are equal in numerical value, the discriminant of ψ, Δ_ψ, vanishes, and Lagrange's theorem can no longer be applied to the function ψ.* Along with Lagrange's theorem all its consequences cease to hold.

It is to *Galois'* genius that we owe a way out of these difficulties which seem to endanger most of our theorems on unsymmetric functions the moment we pass from the general equation to a special equation.†

III.

Netto's exposition of Galois' theory of the *group of an equation,* which is given in chaps. IX. and XIV., seems to us to be open to a number of objections all derived from the fact that the difficulties explained in the preceding section are not sufficiently considered. The term "rational function" is used without an explanation concerning the nature of the coefficients; no distinction is made between formal and numerical invariance; and Lagrange's theorem is used without a previous examination of the discriminant. All this gives to the deductions a certain vagueness and ambiguity. Moreover some of the most important theorems are omitted, others left without a proof. We think it therefore necessary to enter into a detailed analysis and to add such explanations as we believe indispensable to complete Netto's developments.

1) *Definition of the group of an equation.*

Netto gives, in the present edition, a new definition of the group of an equation, which takes a position intermediate between Jordan's and Kronecker's definitions. It is based on the following considerations: Let

$$(1) \qquad f(x) \equiv x^n - c_1 x^{n-1} + c_2 x^{n-2} - \ldots \pm c_n = 0$$

be an equation of the nth degree whose roots $x_1, x_2, \ldots x_n$ are all distinct. The $n!$-valued function

$$(2) \qquad \xi = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

where the α's are arbitrary constants,‡ satisfies a resolvent equation of degree $n!$:

* See NETTO-COLE, § 111.

† See the remarks at the end of the following section.

‡ This is too vague. The α's must be either rational functions with integral coefficients of $c_1, c_2, \ldots c_n$, so chosen that the $n!$ values of ξ are numerically distinct (GALOIS), or else undetermined quantities (KRONECKER). The developments would, however, have gained in clearness as well as in generality if the general idea of domain of rationality had been introduced from the outset, instead of only in chap. XIII.

$$(3) \qquad F(\xi) \equiv \xi^{n!} - A_1\xi^{n!-1} + \ldots \pm A_{n!}$$
$$\equiv \Pi\,[\xi - (\alpha_1 x_{h_1} + \alpha_2 x_{h_2} + \ldots + \alpha_n x_{h_n})],$$

where the product extends over all the $n!$ permutations $h_1, h_2, \ldots h_n$ of the indices $1, 2, \ldots n$, and the coefficients A are rational integral functions* of those of (1) and of $\alpha_1, \alpha_2, \ldots \alpha_n$.

If the coefficients $c_1, c_2, \ldots c_n$ of (1) are entirely independent, $F(\xi)$ cannot break up into rational factors. But for particular values of the coefficients c, $F(\xi)$ may break up into *irreducible* factors with rational coefficients

$$(4) \qquad F(\xi) = F_1(\xi)F_2(\xi) \ldots F_n(\xi).$$

Consider any one of these irreducible factors, say $F_i(\xi)$. It may be written in two different forms: either decomposed into its linear factors

$$(5) \qquad F_i(\xi) = \Pi\,[\xi - (\alpha_1 x_{i_1} + \alpha_2 x_{i_2} + \ldots + \alpha_n x_{i_n})],$$

where the product extends over certain permutations $i_1, i_2, \ldots i_n$ of the indices $1, 2, \ldots n$; or arranged according to powers of ξ

$$(6) \qquad F_i(\xi) = \xi^r - B_1\xi^{r-1} + \ldots \pm B_r.$$

Since $F_i(\xi)$ is supposed to be a rational divisor of $F(\xi)$, the coefficients $B_1, B_2, \ldots B_r$ are expressible as rational functions of $c_1, c_2, \ldots c_n$ and $\alpha_1, \alpha_2, \ldots \alpha_n$.

The "expression" (5) of $F_i(\xi)$ is an *unsymmetric* function of $x_1, x_2, \ldots x_n$, and belongs to a certain group G_i; the "expression" (6) of $F_i(\xi)$ is a *symmetric* function of $x_1, x_2, \ldots x_n$, belongs to the symmetric group and is rationally known.†

From this double expression of $F_i(\xi)$, it follows that every function belonging to G_i is rationally known, being a rational function of $F_i(\xi)$.‡

Netto next shows that the groups G_i all coincide:

$$(7) \qquad G_1 = G_2 = \ldots = G_r,$$

* With integral coefficients. Rational function is here always understood in this sense.

† This is an illustration of the remark 2, a) of the last Section, and shows again that the term "group of the function F_i" is not unambiguous.

‡ This conclusion supposes $\Delta_{F_i} \neq 0$; it is, however, not difficult to prove that this condition is always satisfied, owing to the undetermined quantity ξ, which is contained in F_i.

say $= G$. And it is this group G which Netto defines as *the group of the equation* (1). It is interesting to compare this definition with Jordan's definition on the one hand, and with Kronecker's definition on the other hand.

a) *Jordan's definition.*

Let $F_1(\xi)$ denote that irreducible factor of $F(\xi)$ which admits the root

$$\xi_1 = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n ;$$

all the roots of $F_1(\xi)$ are derived from ξ_1 by the application of certain substitutions

$$(8) \qquad 1, a, b, \ldots l$$

between the letters $x_1, x_2, \ldots x_n$, and $F_1(\xi)$ may be written

$$F_1(\xi) = (\xi - \xi_1)(\xi - \xi_a)(\xi - \xi_b) \ldots (\xi - \xi_l).$$

Jordan proves* the theorem :

The substitutions (8) *form a group, say* G'.

This group G' is the group of the given equation. Starting from this definition, it is easy to show † that Netto's group G is identical with G'. Netto himself uses later on (§229) the identity of both groups,‡ but he omits to prove that the substitutions (8) form a group.

b) *Kronecker's definition.*

Kronecker § considers the irreducible factor $F_1(\xi)$ not as a function of the roots but *as a function of the parameters* α_1, $\alpha_2, \ldots \alpha_n$, which he regards as undetermined quantities. By this ingenious device, he avoids the ambiguity which attaches to the term "group of a function" in the case of non-independent quantities. No matter in which form we may express the function $F_1(\xi)$ it always belongs to one and the same group, say Γ, between $\alpha_1, \alpha_2, \ldots \alpha_n$.

If now we replace, in the group Γ, the letter α by the letter x, we obtain precisely the group G of the equation (1).‖ Netto touches Kronecker's definition in §226, page 267, but without showing its identity with his own definition.

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* JORDAN, l. c. No. 354, and SERRET, l. c. No. 584, Corollaire.

† JORDAN, l. c. No. 351.

‡ There is a similar silent assumption in the proof of (7).

§ "Grundzüge einer arithmetischen Theorie der algebraischen Grössen," §11 ; *Journal für Mathematik*, vol. 92.

‖ For a direct proof of this statement, I refer to my note "Ueber Kronecker's Definition der Gruppe einer Gleichung," in one of the forthcoming numbers of the *Mathematische Annalen*.

2) *Properties of the group of an equation.*

Netto formulates Galois' fundamental theorem as follows:

"*Every function belonging to G is rationally known; and, conversely, every rationally known function belongs to G.*"

We object to the use of the word "belonging to G" without an additional explanation whether it refers to formal or to numerical invariance; the more so, as from all that precedes it seems more likely that it is used in the former sense. And yet, in this case, the above formulation would not give the full meaning of Galois' theorem; the first part would state too little, the second too much.

Illustration: The group of the equation

$$x^4 + 1 = 0$$

with respect to the domain $\mathfrak{R} = 1$ is

$$G = [1\,;\ (12)\ (34)\,;\ (13)\ (24)\,;\ (14)\ (23)].$$

Using the same notation as on page 95 we have

$$x_1x_2^2 + x_3x_4^2 = 0.$$

Hence the expression $x_1x_2^2 + x_3x_4^2$ is rationally known; nevertheless it is not formally unchanged by G.

Galois himself expressly states in a footnote to his proposition I., that he means *numerical* invariance; and, in fact, the whole emphasis of the theorem lies on the numerical invariance.

To complete Netto's proof of the first part of the theorem, the following lemma must be added: "Every function which is numerically unchanged by the substitutions of a group, can always be thrown into such a form that it is also formally unchanged." For if $1, a, b, \ldots l$ are the substitutions of the group, and r their number, then it follows from $\varphi_1 = \varphi_a = \varphi_b = \ldots = \varphi_l$ that

$$\varphi_1 = \frac{1}{r}\,[\varphi_1 + \varphi_a + \varphi_b + \ldots + \varphi_l].$$

As to the second part of Galois' theorem, Netto gives no proof of it either in §153 or in §226. And yet the second part is just as important as the first and by no means self-evident.

In this connection we must mention a statement in §154: "It is clear that every unsymmetric equation $\varphi(x_1, x_2, \ldots x_n) = 0$ between the roots produces an affect." This is only true if $\Delta_\varphi \neq 0$.

Illustration: $x^3 - 2 = 0$. Its roots

$$x_1 = \sqrt[3]{2},\ x_2 = \omega\sqrt[3]{2},\ x_3 = \omega^2\sqrt[3]{2},\ \left(\omega = e^{\frac{2\pi i}{3}}\right)$$

satisfy the unsymmetric relations

$$x_1^2 - x_2x_3 = 0,\ x_2^2 - x_3x_1 = 0,\ x_3^2 - x_1x_2 = 0.$$

Nevertheless it is easily shown that the group of the equation with respect to the domain $\mathfrak{R} = 1$, is the symmetric group; that is, the equation has no affect.

We regret that Netto omits to point out one feature of Galois' theory which seems to us one of the most important, the *reconstruction* of the theorems on rational functions of the roots,* alluded to at the end of the preceding section. This reconstruction may be condensed in a simple practical rule: To pass from a theorem on rational functions of n independent quantities to the corresponding theorem in the case of a special equation, replace "symmetric group" by "group G of the equation" and "formally unchanged" by "numerically unchanged."

Thus the theorem (§29): "Those substitutions (viz., of the symmetric group) which leave a rational function of n independent quantities (formally) unchanged form a group," now takes the form:

Those substitutions *of the group G* which leave a rational function (φ) of the roots *numerically* unchanged form a group (H).†

Example: The group of the equation

$$x^4 + 1 = 0$$

with respect to the domain of rationality $\mathfrak{R} = 1$, is the group

$$G = [1;\ (12)\ (34);\ (13)\ (24);\ (14)\ (23)].$$

Those substitutions of G which leave the function $\varphi = x_1^2 = x_3^2 = x_2x_4$ numerically unchanged, are: 1; (13) (24); they form indeed a group.

Similarly, the propositions of §41 and §53 have to be replaced by the following: If the order of G is ν times the

* See KLEIN, "Vorlesungen über das Ikosaeder," pp. 85, 86; HÖLDER, l. c. §15-20; and my paper, "On the theory of substitution groups" etc. §13, *American Journal of Mathematics*, vol. 13.

† JORDAN, l. c. No. 362. It seems natural to call this group H "the group of φ."

order of H, then the function φ, on being operated upon by all the substitutions of G, takes exactly ν *numerically distinct values*

$$\varphi_1, \varphi_2, \ldots \varphi_\nu .$$

They are the roots of an equation of degree ν, whose coefficients are rationally known, and which, moreover, is *irreducible* in the domain of rationality under consideration.*

Again, Lagrange's theorem takes the form :

If a rational function φ of the roots remains numerically unchanged by all those substitutions of the group G which leave another function ψ numerically unchanged, then φ is rationally expressible in terms of ψ ; and the theorem is true without any exception.†

These propositions contain the complete solution of the difficulties of the preceding section.

3) *Reduction of the group by adjunction.*

Suppose we had found, by solving an auxiliary equation $g(z) = 0$, an irrational function z of the known quantities $\mathfrak{R}', \mathfrak{R}'', \ldots$; we may then, henceforth, consider also z as a known quantity, or, in the language of Galois and Kronecker, *adjoin it to our domain of rationality.*

It may happen that the factors $F_1(\xi), F_2(\xi), \ldots$ of $F(\xi)$, irreducible in the original domain $(\mathfrak{R}', \mathfrak{R}'', \ldots)$, are reducible in the new domain $(z ; \mathfrak{R}', \mathfrak{R}'', \ldots)$. In this case the group *G of $f(x) = 0$ is reduced to a subgroup of G* by the adjunction of z. And the solution of an equation by a chain of auxiliary equations consists, from Galois' point of view, in the successive reduction of its group until the group finally only contains the one substitution 1.

Two cases have to be distinguished according as z is rationally expressible in terms of the roots of the given equation ("natural irrationality") or not ("accessory irrationality," Klein).

The principal theorem concerning the *adjunction of natural irrationalities* is Galois' proposition IV.‡: "By the adjunction of the numerical value of a rational function φ of the roots, the group G of the equation is reduced precisely to that subgroup H of G which leaves φ numerically unchanged." This proposition, which is omitted in Netto's exposition, would have simplified the developments of §§ 230–234 concerning the group Γ of the auxiliary equation $g(\varphi) = 0$, satisfied by the rational function φ. If, in particular, H is a self-conjugate subgroup of G, the group Γ is the quotient-group $G : H$.

* See JORDAN, l. c. No. 366.
† See JORDAN, l. c. No. 362, Cor. II., and KLEIN, l. c.
‡ GALOIS, l. c. p. 41, SERRET, l. c. No. 583, and JORDAN, l. c. No. 362.

Further, if H is a maximal self-conjugate subgroup, the group Γ is simple. Hence the solution of a composite * equation can be reduced, by the successive adjunction of a series of rational functions of the roots, to the solution of a chain of simple regular * equations.†

The *adjunction of accessory irrationalities* is treated in §§236–238. Galois' proposition III., concerning the adjunction of all the roots of an auxiliary equation, is given, and Jordan's theorem concerning the mutual adjunction of all the roots of $g(z) = 0$ to the equation $f(x) = 0$ and of all the roots of $f(x) = 0$ to the equation $g(z) = 0$, with the corollary: "If the group of $f(x) = 0$ is reduced by the solution of a simple equation $g(z) = 0$, then the roots of the latter equation are rational functions of the roots of $f(x) = 0$."

This corollary contains, as Netto points out, a proof and an extension of Abel's celebrated theorem concerning the irrationalities which enter into the solution of an algebraically solvable equation.

IV.

The three following chapters are devoted to three important classes of special equations: cyclotomic equations, Abelian equations, and equations with rational relations between three roots.

The *cyclotomic equation*

$$\frac{x^p - 1}{x - 1} \equiv x^{p-1} + x^{p-2} + \ldots + x + 1 = 0$$

for a prime number p is irreducible in the domain $\mathfrak{R} = 1$; if ω denotes one of its roots, all the roots may be written

$$\omega^g,\ \omega^{g^2},\ \cdots\ \omega^{g^{p-1}}$$

g denoting a primitive root (mod p). Hence it follows by applying the second part of Galois' fundamental theorem, that the group G of the equation is the cyclic group consisting of the p powers of the substitution

$$\left(\omega^g,\ \omega^{g^2},\ \cdots\ \omega^{g^{p-1}}\right).$$

* An equation is called simple, composite, regular, etc., if its group is simple, composite, regular, etc.

† Compare JORDAN, l. c. No. 362, 366–372, and HÖLDER, l. c. §§18–20.

The $(p-1)$th power of Lagrange's expression

$$(\alpha, \omega) = \omega + \alpha\omega^{\theta} + \alpha^2\omega^{\theta^2} + \cdots \alpha^{p-2}\omega^{\theta^{p-2}},$$

where α denotes a primitive root of the equation $x^{p-1} - 1 = 0$, is unchanged by the substitutions of G and therefore rationally expressible in terms of α. Hence the solution of the cyclotomic equation for the prime number p requires only the determination of a primitive root of the equation $x^{p-1} - 1 = 0$, and the extraction of the $(p-1)$th root of an expression which is then rationally known.

If $p_1, p_2, \ldots$ are the prime factors of $p - 1$, the solution can be decomposed into a chain of binomial equations of degrees $p_1, p_2, \ldots$ Hence Gauss's celebrated theorem: If $2^m + 1$ is a prime number, the regular polygon of $2^m + 1$ sides can be constructed by means of ruler and compass. The constructions for $p = 5$ and $p = 17$ are given in full detail.

The chapter on *Abelian equations* begins with a reproduction of Abel's researches on irreducible equations of which one root x'_1 is a rational function of another root x_1: $x'_1 = \theta(x_1)$. The group of such an equation is determined and found to be non-primitive. If in particular all the roots can be arranged in one cycle

$$x_1, \theta(x_1), \theta^2(x_1), \ldots \theta^{m-1}(x_1); \quad \theta^m(x_1) = x_1,$$

the group is cyclic, and we have the immediate generalization of the cyclotomic equations ("simplest Abelian equations"), and the equation is solvable by radicals. Two examples of this type of equations are discussed; in the one case

$$\theta(x) = \frac{\alpha x + \beta}{\gamma x + \delta};$$

in the other, which is a new addition of the English edition, $\theta(x)$ is an integral function.

Then follows the general definition of Abelian equations in accordance with Jordan: an equation is called Abelian if all its roots are rational functions of one of them

$$x_1, \theta_1(x_1), \theta_2(x_1), \ldots \theta_{n-1}(x_1)$$

and besides the operations θ are commutative:

$$\theta_\alpha\theta_\beta(x_1) = \theta_\beta\theta_\alpha(x_1).$$

The substitutions of the group of an Abelian equation are all *commutative;* conversely, if the substitutions of the group

of an equation are all commutative, the equation is an Abelian equation.

Kronecker's as well as Jordan's treatment of Abelian equations are developed; both methods lead to the theorem due to Abel: *Every Abelian equation is solvable by radicals.* As an illustration the equation for $\cos \frac{2\pi}{n}$ is discussed.

In chap. XII. irreducible equations are considered all the roots of which are rational functions of two among them. If, in particular, the degree is a prime number p, the equation is a *Galois equation;* its group is the metacyclical group or one of its transitive subgroups. The solution of a Galois equation reduces to that of two Abelian equations. The binomial equation of prime degree $x^p - A = 0$ is the simplest example of a Galois equation.

An allied class of equations are Nöther's *triad equations,* the theory of which is developed in the second part of the chapter; a well-known example of a triad equation is the equation of the 9th degree for the determination of the nine points of inflection of a plane curve of the third order. An interesting investigation on triad equations of degree 7 has been added in the new edition. The group of the most general irreducible triad equation of degree 7 is the "Kronecker group" of order 168 defined by

$$| z \;\; az + b |, \; | z \;\; a\theta(z + b) + c |$$

$$[a = 1, 2, 4; \; b, c = 0, 1, \ldots 6; \; \theta(z) = -z^2(z^3 + 1)].$$

The two remaining chapters are devoted to the general theory of *algebraically solvable equations,* treating, however, the problem by two entirely different methods, which may be characterized as *Abel's* and *Galois'*.

1) *Abel's Method* (chap. XIII.).

The first part of the chapter reproduces Abel's proof, simplified by Kronecker, of the fundamental theorem that the solution of an algebraically solvable equation can always be performed by a chain of binomial equations of prime degrees whose roots are rationally expressible in terms of the roots of the given equation and of certain roots of unity.*

Combining this proposition with the results of chap. III. concerning the existence of rational functions of n undeter-

* The shorter form in which the above theorem is given in Art. 85 of my paper "On the theory of substitution groups, etc.," *American Journal,* vol. 13, viz.: "The radicals which enter into the solution of a solvable equation are always rationally expressible in terms of the roots and of certain roots of unity," is not exact.

mined quantities, a power of which is symmetric or two-valued, it follows at once that *the general equations of a degree higher than the fourth are not algebraically solvable.*

The latter part of the chapter is devoted to the difficult problem of the *explicit expression of the roots of a solvable equation.* Abel had obtained in his memoir "Sur la résolution algébrique des équations," of which, unfortunately, we possess but fragments, two expressions for a root of an algebraically solvable equation of prime degree p: The first is

$$g_0 + \sqrt[p]{R} + \sqrt[p]{R_2} + \ldots \sqrt[p]{R_{p-1}}, \tag{1}$$

where g_0 is a rational quantity * and $R_1, R_2, \ldots R_{p-1}$ are the roots of an equation of degree $p - 1$ whose coefficients are rational quantities. The second is

$$g_0 + s^{\frac{1}{p}} + \varphi_2(s)\, s^{\frac{2}{p}} + \ldots + \varphi_{p-1}(s)\, s^{\frac{p-1}{p}}, \tag{2}$$

where $\varphi_2, \ldots \varphi_{p-1}$ are rational functions whose coefficients are rational quantities. These two forms are necessary but not sufficient for a root of a solvable irreducible equation of prime degree.

Kronecker, completing Abel's researches and generalizing a result obtained by Abel, for $p = 5$, gave,†—yet without a proof,—the further conditions which must be satisfied by the quantities $R_1, R_2, \ldots R_{p-1}$, in order that the expression (1) may actually satisfy an irreducible solvable equation of degree p.

Netto not only proves Abel's results, but—and this is one of the most important additions of the new edition—he also gives a proof and further development of Kronecker's propositions.

2) *Galois' Method* (chap. XV.).

Galois' method consists in the successive reduction of the group of the given equation by the successive adjunction of the various radicals which enter into the expression of the roots. By means of his propositions III. and IV., mentioned in section III., Galois establishes the fundamental theorem:

In order that an equation may be algebraically solvable, it is necessary and sufficient that all the factors of composition of its group be prime numbers.

Hence arises the problem: To determine all groups of sub-

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* That is, rationally expressible in terms of $\mathfrak{R}'$, $\mathfrak{R}''$, . . .

† *Monatsberichte der Berliner Akad.* 1853, translated in SERRET, Cours d'Algèbre supérieure, 4me édition, No. 599.

stitutions between n letters whose factors of composition are all of them prime numbers.

The problem can be reduced to the case of transitive groups. A further reduction results from the following propositions:

Every equation the group of which is non-primitive, is the result of the elimination of an auxiliary quantity y from two irreducible equations,

$$y^\nu - A_1 y^{\nu-1} + \ldots \pm A_\nu = 0,$$

$$x^m - S_1(y)x^{m-1} + \ldots \pm S_m(y) = 0,$$

(the A's being rational quantities and the $S(y)$'s rational functions of y); and

If the degree of a solvable irreducible equation is divisible by two different prime numbers, its group is non-primitive.

We may therefore confine ourselves to the consideration of primitive equations whose degree is a power of a prime, $n = p^k$; for these the following theorem holds:

The group of every solvable primitive equation of degree p^k consists of the group of the arithmetic substitutions of the degree p^k,

$$| z_1, z_2, \ldots z_k \quad z_1 + \alpha_1, z_2 + \alpha_2, \ldots z_k + \alpha_k | \pmod p,$$

combined with geometric substitutions of the same degree,

$$| z_1, z_2, \ldots z_k \quad a_1 z_1 + b_1 z_2 + \ldots + c_1 z_k, \quad a_2 z_1 + b_2 z_2 + \ldots + c_2 z_k, \quad \ldots | \pmod p.$$

If $k = 1$, the converse of this theorem is also true; hence Galois' theorem:

The group of a solvable irreducible equation of prime degree is the metacyclic group (or one of its transitive subgroups); and conversely, every metacyclic equation is algebraically solvable.

If, on the contrary, $k > 1$, not every group of the above form belongs to a solvable equation; and the determination of all solvable groups is, in this case, as yet an unsolved problem.

These are the principal subjects treated in the last chapter of Netto's book.

O. BOLZA.

UNIVERSITY OF CHICAGO, *January*, 1893.

A BIT OF MATHEMATICAL HISTORY.

BY DR. MAXIME BÔCHER.

In the transactions of the Academy of St. Petersburg for the year 1764 is a memoir by Leonhard Euler entitled: *De Motu Vibratorio Tympanorum*, which is remarkable in several respects. Certainly the last part of this memoir has been lost sight of for many years by mathematicians. Whether the same can be said of the first part I do not know,* but I will venture to give a brief account of the contents of the whole paper.

During the second third of the eighteenth century a number of mathematicians, among whom Daniel Bernoulli and Euler deserve special mention, had given much attention to problems concerning the vibration of such one dimensional bodies as strings, rods and columns of air, but the paper now under consideration is the first in which a two dimensional problem of the same nature was taken up; the question chosen being the problem of the vibration of a stretched elastic membrane. The now familiar equation of motion of the membrane:

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

is first obtained, and then follows a brief discussion of the simple harmonic vibrations of a rectangular membrane, but although the square membrane is chosen as a simple case in which to give numerical results the peculiar interest of this case is not mentioned.

Failing to find a general method for the discussion of the vibrations of membranes of *any* shape, Euler next takes up the question of the vibration of a circular membrane. The first step towards the solution of this problem is the introduction of polar in place of rectangular coördinates in the equation of motion, and even this step is worthy of notice as being apparently the first case of a change of independent variables in a partial differential equation.†

* Certainly Poisson did not know of the existence of this paper when he wrote his celebrated memoir on elasticity (*Mémoires de l'Institut*, 1829) which is frequently referred to as containing the first investigation of the vibration of membranes.

† Euler himself says: Modus autem, quo hanc æquationem elicuimus, novam quandam algorithmi speciem constituit, quæ omni attentione digna videtur.

Euler then proceeds to find an infinite number of solutions of this equation in the form of products of three factors, functions respectively of the three independent variables; a step which although perhaps small in itself is important as being in the line of development of a great theory.

One of these three factors turned out of course to be what is now known as a Bessel's function,* and the fact that these functions occur in this paper is the most important point I have to bring out in this note. Until recently it was thought that Fourier had introduced the Bessel's function of the zeroth order (1812), and that the other functions of integral orders were first made use of by Bessel (1818). A few years ago, however, Maggi called attention to papers by Daniel Bernoulli (1732) and Euler on the vibration of a heavy string suspended from one end, in which Bessel's functions of the zeroth order occur.† That further than this *all* the Bessel's functions of the first kind and of integral orders go back like so many other things to the great mathematicians of the eighteenth century seems to have remained unnoticed.

The second solution of Bessel's equation is known to contain a logarithmic term. Euler, however, appears to find a second solution free from this term. The explanation of this peculiar mistake is that Euler's second solution really vanishes identically.‡ Fortunately he decides from quite different considerations that this second solution is not applicable to the problem in hand.

Finally I will quote in full a passage by the same author which, although not occurring in the paper just discussed, bears upon the same questions. It will be found in the transactions of the Academy of St. Petersburg for 1781 (p. 164) in a paper on the above mentioned problem of the vibration of a heavy string hanging from one end. The expression $\Pi : \frac{a}{b}$ which occurs in the first line is essentially a Bessel's function of the first kind and zeroth order:

"Quodsi ergo æquationis $\Pi : \frac{a}{b} = 0$ omnes radices assignare liceret, ex iis utique solutionem tam generalem deducere possemus, quæ sine ullo dubio omnes plane motus, qui in fune

* Euler mentions the connection between Bessel's equation and Riccati's equation, and hence determines the cases, which he does not further consider, where it can be solved in finite form.

† For references see HEINE, Handbuch, vol. II., p. 343.

‡ Euler avoids this mistake in a subsequent paper in which he introduces the Bessel's functions of the second kind although only of the zeroth order; see the *Transactions of the Academy of St. Petersburg* for 1781, p. 187, where a problem similar to Bernoulli's problem of the swinging string is considered.

locum habere queant, in se complectatur. Neque vero idcirco problema principale resolvere liceret, quo pro statu initiali quocunque ejus motus secuturus requiritur; ad hoc enim necesse esset, infinitas illas constantes arbitrarias pro statu initiali dato debite determinare, quod certe opus omnes vires analyseos longe esset superaturum."

In order to see how remarkable this passage is it is necessary to recall the celebrated triangular contest between D'Alembert, Euler and Daniel Bernoulli which began at about the middle of the eighteenth century.* We have here to deal with the question under discussion between Bernoulli and Euler, the former maintaining that the most general vibration of a stretched string could be expressed as a trigonometric series, the latter that this could give only a particular class of solutions, since (a fact no one doubted at that time) a function which would be graphically represented by a broken line could not possibly be expressed as a trigonometric series, or in fact by any analytic formula.

Now the above quotation refers to an entirely analogous problem, the principal difference being that the trigonometric series is replaced by a series involving Bessel's functions, and here we find Euler taking Bernoulli's position and asserting that without any doubt this series expresses the most general motion possible! I am at a loss to explain this passage, for we must apparently either assume that Euler had forgotten the previous discussion which had lasted so long and formed such an important episode in his life and in which, considering the mathematical knowledge of the time, his position had been unassailable; or that he had anticipated Fourier's wonderful idea that *any* function can be developed in a trigonometric series, or a series involving Bessel's functions. That he should have arrived at this point without the method of determining the coefficients seems incredible.

HARVARD UNIVERSITY, *January*, 1893.

NOTES.

THE annual meeting of the NEW YORK MATHEMATICAL SOCIETY was held Thursday afternoon, December 29, at four o'clock, the president, Dr. McClintock, in the chair. The council announced that Professor Henry B. Fine had been appointed to act as delegate of the Society to the meetings of the American Philosophical Society on the occasion of the

* See RIEMANN, Works, pp. 214–218 (sec. ed. pp. 227–231).

150th anniversary of its founding. Reports were presented by the secretary, treasurer and librarian. The secretary stated that the membership of the Society was 234, of whom 44 resided in New York city and the immediate vicinity. The average attendance of members at the meetings had been 15 or 16, and the average total attendance 18. The librarian stated that in the library there were 36 volumes and 54 pamphlets, and that 15 journals had been placed on the exchange list of the Society. The treasurer's report having been read, an auditing committee was appointed to examine his accounts.

The committee on nominations elected at the preceding meeting, reported a ticket for the officers and other members of the council for the ensuing year, and, a ballot being taken, this ticket was unanimously elected. The officers and council for 1893, are :—President, Dr. Emory McClintock ; Vice-President, Professor Henry B. Fine ; Secretary, Dr. Thomas S. Fiske ; Treasurer, Mr. Harold Jacoby ; Librarian, Professor D. A. Murray ; Committee of Publication, Dr. Thomas S. Fiske and Mr. Harold Jacoby ; other members of the Council, Professor Thomas Craig, Professor W. Woolsey Johnson, Professor J. E. Oliver, Professor J. K. Rees, Professor J. H. Van Amringe.

Professor M. I. Pupin communicated an informal note on a proof of the principal property of two perspective triangles, viz., that the points in which the corresponding sides intersect are all on the same straight line. If the triangles are plane sections of a triangular pyramid, it is sufficient to point out that these points must lie on the line of intersection of their planes. To prove the property for two perspective triangles in the same plane, it is necessary only to show that they are projections of plane sections of a triangular pyramid. The converse proposition also follows at once by the method of projections.

The same method of demonstration may be applied to show that the corresponding sides of two perspective polygons intersect in points which are all on the same straight line provided that these polygons represent the projections of plane sections of some pyramid. If the corresponding sides of two polygons intersect on the same straight line without making the two polygons perspective to each other, they may be taken as representing the projections of plane sections of wedge-shaped bodies. To this class of figures belong any two closed funicular polygons corresponding to the same system of forces. (Compare Minchin's *Statics*, p. 105.) The proper qualification of the results mentioned in the latter portion of the note, gave rise to some discussion in which Professors Webb and Merriman took part.

REPORT OF THE TREASURER FOR THE YEAR 1892.

Receipts.		*Expenditures.*	
Balance from 1891.....	$271.00	Printing list of members...	$98.00
Net receipts from members, subscribers, and advertising.....	1179.70	" Bulletin.....	793.68
		Stationery, postage, distribution of Bulletin, and miscellaneous.....	214.68
		Balance.....	344.34
	$1450.70		$1450.70

HAROLD JACOBY, *Treasurer.*

We have examined the treasurer's accounts, and found the same correct.

J. H. VAN AMRINGE, J. K. REES, W. S. DENNETT, } *Auditing Committee.*

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, January 7, at half-past three o'clock, the president, Dr. McClintock, in the chair. A paper by Professor W. H. Echols, "On a general formula for the expansion of functions in series," was read by the secretary. This paper may be regarded as an exposition of a general theorem which is the basis of a series of papers in course of publication in the *Annals of Mathematics* under the title "On certain determinant forms and their applications." T. S. F.

THE council of the German mathematical association (*Deutsche Mathematiker-Vereinigung*) has issued a preliminary report, from which it appears that the membership is at present 215. The annual meeting will be held at Munich, in the month of September, during the week preceding the *Naturforscher-Versammlung;* the exact date will be announced later.

As last year's meeting had to be abandoned on account of the cholera, the exhibition of mathematical models and instruments planned for that meeting will be held this year at Munich, in the rooms of the Polytechnic School; it is to be open during the whole month of September and promises to be of unusual interest. Prof. W. Dyck, in connection with many others, had undertaken the preparation of a descriptive and illustrative catalogue for the proposed exhibition of 1892. This catalogue, a volume of about 450 pages, with numerous illustrations, is now ready, and can be obtained directly from Prof. W. Dyck (München, Hildegardstrasse 1½), at the price of mk. 9.80 (including postage). While this volume can of course not be regarded as a complete catalogue of the coming

exhibition, it will serve to show the range and importance of the work to be accomplished.

The objects to be exhibited are classified under three main heads which we may briefly describe as I. analysis, II. geometry, III. mechanics. Class I. is subdivided as follows: (1) arithmetic, comprising calculating machines and apparatus used in the theory of probability; (2) algebra and theory of functions, where, in addition to models and drawings, we find mechanical devices for solving equations and constructing the dependence of functions on their variables; (3) integral calculus, with instruments for rectifying curves, measuring areas, etc. In the class devoted to geometry we have, besides drawing instruments, a large variety of models and graphical illustrations exhibiting the properties of geometrical solids, surfaces, and curves. Class III. has three principal divisions. In the first, mechanics proper, are comprised (a) models and instruments for demonstrating dynamical propositions, and (b) kinematical models and instruments, with applications to machinery (a very extensive division). In the second division, mathematical physics, we find (a) apparatus illustrative of the wave-theory; (b) and (c) models illustrating the structure and physical properties of crystals; (d) thermodynamical models and drawings; (e) apparatus for the illustration of electrodynamic phenomena. The third division, technical applications, contains geodetic, nautical and meteorological instruments.

This description of instruments is preceded by a series of monographs and notes covering about 135 pages. The authors and subjects are as follows: *F. Klein*, Geometrical illustration of the enumeration of the real roots of algebraic equations; *A. Voss*, On systems of equidistant curves on curved surfaces; *A. Brill*, On the reduction of the higher singularities of an algebraic curve to elementary singularities; *G. Hauck*, On the geometric postulates of solid geometry in their relation to the methods of descriptive geometry; *A. von Braunmühl*, Historical sketch of the organic generation of plane curves from the oldest times to the end of the eighteenth century; *L. Boltzmann*, On the methods of theoretical physics; *A. Amsler*, On mechanical integrations; *O. Henrici*, On instruments for harmonic analysis. A. Z.

R. S. WOODWARD, C.E., Ph.D. (University of Michigan), of the U. S. Coast Survey, Washington, D. C., has been appointed Professor of Mechanics at Columbia College, the appointment to take effect on July 1.

THE nine conferences on the curriculum of the secondary schools, appointed by the National Educational Association to

be held at various places throughout the United States, met on December 28, 1892. That on Mathematics met at Cambridge, Mass. The members of the committee were: Professor W. E. Byerly, Harvard University; Professor Florian Cajori, Colorado College; Chancellor W. S. Chaplin, Washington University; Mr. Arthur H. Cutler, New York; Professor H. B. Fine, Princeton College; Mr. W. A. Greeson, Grand Rapids, Mich.; Professor Simon Newcomb, Johns Hopkins University; Mr. J. L. Patterson, Lawrenceville, N. J.; Professor G. D. Olds, Amherst College; Professor T. H. Safford, Williams College. It was the unanimous opinion of the committee that important changes are needed. Its proceedings, however, can not be made public until laid before the general committee of ten in April or May. H. J.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ECKHARDT (E.). Ein Rotationsproblem. Die Dreitheilung des Winkels. Die Darstellung der Wurzeln der Gleichungen dritten Grades durch Zeichnung. Marburg 1892. 8vo. 56 pp. 2 plates. Mk. 1.80

HAYWARD (R. B.). The algebra of coplanar vectors and trigonometry. London, Macmillan, 1892. 8vo. 376 pp. 8s. 6d.

JAHRESBERICHT der deutschen Mathematiker-Vereinigung. Band I: 1890-91. Enthaltend Chronik für 1890-91, Bericht über die in Halle 1891 gehaltenen Vorträge und ausführlichen Bericht über die Fortschritte der projectiven Invariantentheorie im letzten Vierteljahrhundert, von Dr. W. F. Meyer. Herausgegeben von G. Cantor, W. Dyck, E. Lampe. Berlin, Reimer, 1892. 8vo. 292 pp. Mk. 7.60

LAISANT (C. A.). Recueil de problèmes de mathématiques, classés par divisions scientifiques, contenant les énoncés avec renvoi aux solutions de tous les problèmes posés, depuis l'origine, dans divers journaux: Nouvelles annales de math., Journal de math. élém. et de math. spéc., Mathésis, Nouvelle correspondance math. (In 7 vols.) Paris. Gauthier-Villars. 8vo.

Vol. IV. Géométrie analytique à deux dimensions (et géométrie supérieure). 1893. Fr. 6.00

Vol. V. Géométrie analytique à trois dimensions (et géométrie supérieure). 1893. Fr. 2.50

MOLIEN (T.). Über Systeme höherer complexer Zahlen. (Dissertation.) Dorpat, Karow, 1892. 8vo. 74 pp. Mk. 2.00

MÜLLER (M.). Über algebraisch-logarithmische Integrale von Systemen algebraischer Differentialgleichungen. Heidelberg, 1892. 8vo. 64 pp. Mk. 1.50

POINCARÉ (H.). Les méthodes nouvelles de la mécanique céleste. (In 2 vols.) Vol. II: Méthodes de MM. Newcomb, Gyldén, Lindstedt et Bohlin. Paris, Gauthier-Villars, 1893. Fr. 12.00

ROST (G.). Untersuchungen über die allgemeinste lineare Substitution, deren Potenzen eine endliche Gruppe bilden. Leipzig, Teubner, 1892. 4to. 28 pp. Mk. 1.20

ROTHHOLZ (J.). Beiträge zum Fermat'schen Lehrsatz. Giessen, 1892. 8vo. 30 pp. Mk. 1.20

SAALSCHÜTZ (L.). Vorlesungen über die Bernoullischen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen. Berlin, 1893. 8vo. 8 and 207 pp. Mk. 5.00

SCHEFFLER (H.). Die quadratische Zerfällung von Primzahlen. Leipzig, Förster, 1892. 8vo. 3 and 169 pp. Mk. 3.00

SIMON. Leitfaden der analytischen Geometrie der Ebene. Berlin, Weidmann, 1892. 71 pp. Mk. 1.00

SMITH (W. B.). Introductory modern geometry of the point, ray, and circle. Parts I and II. New York, Macmillan, 1892. 8vo. 9 and 297 pp. $1.10

II. ELEMENTARY MATHEMATICS.

BIASI (G.). Elementi di aritmetica e algebra, esposti con metodo sintetico. Sassari, Galizzi, 1892. 224 pp.

BURALI-FORTI (C.). Aritmetica razionale (i numeri razionali). Per gli istituti tecnici, ginnasi superiori e schuole normali. Torino, Petrini, 1892. 112 pp. Fr. 2.25

COMBEROUSSE (C. de). Cours de mathématiques. (In 6 vols.) Vol. II: Géométrie élémentaire, plane et dans l'espace. Trigonométrie rectiligne et sphérique. 3me éd., revue et augmentée. Paris, Gauthier-Villars, 1893. 8vo. fig. Fr. 13.00

ELLIOT (J.). Tables of logarithms. 7th ed. London, Stewart. 8vo. 1s.

EUCLIDE. Gli elementi di Euclide; nuova edizione, modificata ed accresciuta dal Dott. M. Gremigni. Libro I. Firenze, Le Monnier, 1893.

FENKNER (H.). Arithmetische Aufgaben, mit besonderer Berücksichtigung von Anwendungen auf dem Gebiete der Geometrie, Trigonometrie, Physik und Chemie. Braunschweig, Salle, 1893. 8vo. 127 pp. Mk. 2.00

FIORE (V.). Primi elementi di geometria. Napoli, Chiurazzi, 1892. 176 pp. Fr. 2.50

JONES (G. W.). Logarithmic tables. 4th ed., enlarged and reset. Ithaca, Jones, 1893. 8vo. 160 pp. $1.00

KLEYER (A.) and SACHS (J.). Lehrbuch der ebenen Elementar-Geometrie (Planimetrie). Nach System Kleyer bearbeitet. Theil V: Die Flächen der geradlinigen Figuren. Bearbeitet von J. Sachs. Stuttgart, Maier, 1892. 8vo. fig. Mk. 4.00

LAISANT (C. A.) and PERRIN (F.). Premiers principes d'algèbre avec plus de 1200 exercices gradués. Paris, Delagrave, 1892. 10 and 346 pp.

LYLE (J. N.). Euclid and the Anti-Euclidians. St. Louis, Frederick Printing Co.

NAGEL. Lehrbuch der Stereometrie. 5te Aufl., von Schröder. Nürnberg, Korn, 1892.

PORTA (F.). Complementi di matematica. 3za ed., redatta secondo i nuovi programmi. Torino, Bocca, 1892. 12mo. Fr. 3.50

SCHERER. Hilfstafel sur Berechnung rechtwinkliger Coordinaten mit Scherers logarithmisch-graphischer Rechentafel. Kassel, Klaunig, 1892. Mk. 1.00

STAUDACHER (H.). Lehrbuch der combinatorischen Operationen (Permutationen, Combinationen. Variationen). Nach System Kleyer bearbeitet. Stuttgart, Maier, 1892. 8vo. Mk. 6.00

WEHNER (H.). Leitfaden für den stereometrischen Unterricht an Realschulen. Leipzig, Teubner, 1892. 8vo. 8 and 54 pp. fig. Mk. 0.80

WEICHOLD (G.). Lehrbuch der Determinanten und deren Anwendungen. Nach System Kleyer bearbeitet. Stuttgart, Maier, 1892. 8vo. Mk. 10.00

WROBEL. Übungsbuch zur Arithmetik und Algebra, enthaltend die Formeln, Lehrsätze und Auflösungsmethoden in systematischer Anordnung und eine grosse Anzahl von Fragen und Aufgaben. 1ter Teil. 2te Aufl. Rostock, Werther, 1892. 299 pp. Mk. 2.60

III. APPLIED MATHEMATICS.

BAILLAUD (B.). Cours d'astronomie à l'usage des étudiants des Facultés des sciences. (In 2 vols.) Vol. I: Quelques théories applicables à l'étude des sciences expérimentales. Instruments d'optique. Instruments d'astronomie. Paris, Gauthier-Villars, 1893. Fr. 8.00

BASSET (A. B.). A treatise on hydrodynamics. Vol. II. Cambridge, 1892. 8vo. 12s. 6d.

BEDELL (F.) and CREHORE (A. C.). Alternating currents: an analytical and graphical treatment for students and engineers. New York, Johnston, 1892. 8vo. 300 pp. fig. $2.50

BÖRNER (H.). Lehrbuch der Physik für höhere Lehranstalten sowie zur Einführung in das Studium der neueren Physik. Berlin, Weidmann, 1892. 8vo. 12 and 584 pp. fig. Mk. 6.00

BUDDE (E.). Physik des Äthers. Berlin, Reimer, 1892. 820 pp. Mk. 17.00

BURN (R. S.) [editor]. The student's introduction to mechanics; by various writers. New York, Ward & Co, 1893. 8vo. 544 pp. Cloth. $2.00

CHRISTIANSEN (C.). Laerebog i Fysik. Vol. I: Mekanisk Fysik og Varmelaere. Köbenhavn, Philipsen, 1892. 8vo. 8 and 428 pp. Mk. 10.00

COLE (F. N.). The diurnal variation of barometric pressure. Washington, Weather Bureau, 1892. 8vo. 32 pp.

DUFAILLY (C.). Mécanique; nouv. éd. Paris, Delagrave, 1892. 8vo. 144 pp.

GRAF (J. H.). Das Leben und Wirken des Physikers und Astronomen Johann Jacob Huber aus Basel (1733–98). Bern, 1892. 8vo. 75 pp. Portrait and 1 plate. Mk. 1.00

GRAWINKEL (C.) and STRECKER (E.). Hilfsbuch für die Elektrotechnik. 3te, vermehrte und verbesserte Auflage. Berlin, 1893. 8vo. 10 and 642 pp. fig. Cloth. Mk. 12.00

HEAVISIDE (O.). Electrical papers. 2 vols. New York, Macmillan, 1892. 8vo. $10.00

HEYDWEILLER (A.). Hülfsbuch für die Ausführung electrischer Messungen. Leipzig. Barth, 1892. 8vo. 8 and 262 pp.

HOFFMANN (G.). Die Anderssohn'sche Drucktheorie und ihre Bedeutung für die einheitliche Erklärung der physischen Erscheinungen. Halle a. S., Schwetschke, 1892. 8vo. 44 pp. Mk. 1.00

HOVESTADT (H.). Absolute Masse und Dimensionen der physikalischen Grössen. Nach System Kleyer bearbeitet. Stuttgart, Maier. 1892.

JAMET (V.). Traité de mécanique à l'usage des candidats à l'École polytechnique. Paris, Carré, 1892. 8vo. Fr. 5.00

JAMIESON (A.). Elementary manual on applied mechanics. Philadelphia, Lippincott, 1892. 12mo. 266 pp. Cloth. $1.25

KITTLER (E.). Handbuch der Elektrotechnik. (In 3 vols.) Vol. I. 2te Aufl. Stuttgart, Enke, 1892. 8vo. 1008 pp. fig. Mk. 40.00

KLIMPERT (R.). Lehrbuch der Hydrodynamik. Nach System Kleyer bearbeitet. Stuttgart, Maier, 1892.

KOEPKE (E.). Beiträge zur Untersuchung der Bewegung eines schweren Punktes auf einer Rotationsfläche. Halle 1892. 8vo. 46 pp. Mk. 1.20

KRIEMLER (C. J.). Aus der Festigkeitslehre. Der Spannungszustand in den Punkten eines geraden Stabes bei den vier einfachen Fällen der Beanspruchung, dargestellt zur Einführung in das Studium der Festigkeitslehre. Vevey, Roth, 1892. 8vo. 127 pp. 1 plate. fig. Mk. 4.00

LONEY (S. L.). The elements of statics and dynamics. (In 2 parts.) Part I: Elements of statics. 2nd ed., revised. New York, Macmillan, 1892. 8vo. fig. Cloth. $1.25

OHM (S.). Gesammelte Abhandlungen. Herausgegeben und eingeleitet von E. Lommel. Leipzig, Barth, 1892. Mk. 20.00

PARK (H.). Modern mechanism, exhibiting the latest progress in machines, motors, and the transmission of power. London, 1892. 8vo. 920 pp. fig. Cloth. 31s. 6d.

PAULSEN (A. F. W.). Naturkraefterne, deres Love og vigtigste Anvendelser. 2de, omarbejdede og forögede Udgave. Kjöbenhavn 1892. 8vo. Lieferung 10: p. 577–640. Mk. 1.20

PIZZETTI (P.). I fondamenti matematici per la critica dei risultati sperimentali. 8vo. 224 pp. Genova, R. Ist. Sordo-Mutti, 1892.

RAUSENBERGER (O.). Lehrbuch der analytischen Mechanik. 2te, wohlfeile Ausg. in einem Bande. Leipzig, Teubner, 1893. 8vo. 8, 318, 6, and 336 pp. Mk. 8.00

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SOME OF THE DEVELOPMENTS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS BETWEEN 1878 AND 1893.*

BY PROF. T. CRAIG.

SINCE the principles of the infinitesimal calculus were established, the analyst has been confronted by three problems, to wit:

The solution of algebraic equations;

The integration of algebraic differentials;

The integration of differential equations.

The history of these three problems is the same. After long and ineffectual efforts to conduct them to simpler problems, mathematicians have reconciled themselves to study these three great problems for themselves, and have been rewarded by abundant success.

For a long time the algebraist hoped and strove to solve all algebraic equations by aid of radicals. That hope has, however, been abandoned, and to-day the algebraic functions are as well known as the radicals to which it was hoped to conduct them. In the same way the integrals of algebraic differentials, which were long studied with the aim of reducing them to the elementary functions of algebra, to the logarithmic or trigonometric functions, are to-day expressed by the aid of new transcendants; and the elliptic and Abelian functions have as well defined a place in analysis as the logarithmic and trigonometric functions had less than a century ago.

Much the same thing is true of differential equations. The number of equations integrable by quadratures is extremely limited, and before the mathematician had decided to study the integrals for themselves, to study them as *functions* defined by a differential equation, all this analytical domain was but a vast *terra incognita*, which seemed to be forever interdicted to the explorer. Cauchy was the first to penetrate to the interior of this unknown region, which he did by aid of the very ingenious method which he called the calculus of limits. Many others followed him, among whom it suffices for the present to mention Fuchs, Briot and Bouquet, and Madame Kowalevski, all of whom employed his method with success.

Before taking up the class of differential equations with which these and other illustrious names have been particularly identified, it will be desirable to speak of the class of equations which, in simplicity at least, ranks first. These are the linear differential equations.

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* Read before the New York Mathematical Society, February 4, 1893.

From the time of Euler until within recent years, the only class of ordinary linear differential equations for which a general method of integration was known, was that of equations with constant coefficients. These are still the only equations whose purely external form shows that they are integrable. Among these is, of course, included Legendre's equation,

$$(ax+b)^n \frac{d^n y}{dx^n} + A(ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + L(ax+b)\frac{dy}{dx} + My = 0;$$

which by the transformation $ax + b = e^t$ is at once conducted to an equation with constant coefficients.

Since the epoch-making researches of Fuchs, published in *Crelle* in volumes 66 and 68, the systematic and well-directed efforts of the most illustrious mathematicians of the age have succeeded in increasing the number of linear differential equations, for which there exist methods of integration both sure and general. The distinctive characteristics of such equations are not of a nature to be perceived at a first glance as in the cases just mentioned, but can nevertheless be recognized by aid of purely algebraic operations.

The characteristic property of a differential equation of this type is that its general integral is a *uniform* function of the independent variable.

It is by the study of the singular points of the equation that we recognize whether or not this characteristic exists. If it exists, the equation can be integrated. Its general integral is, in fact, the quotient of two synectic functions, of which one, the denominator, can be written down almost at once. This is a polynomial if the singular points are limited in number, and a holomorphic function which can be constructed by Weierstrass's method if there are an infinite number of singular points. The numerator function must be obtained by means of the differential equation. The equation is thus integrated, since its general integral is represented by one and the same function, the quotient of those just mentioned, for all values of the variable. In general this integral is a new function, but in special cases we can express it by aid of the functions already introduced into analysis. These cases are two in number. The first case, studied by Halphen, is when the general integral is uniform not only for all finite values of the variable, but for infinite values. When this is the case, the integral is a rational function. This result can only hold for equations with rational coefficients. The second case was suggested by investigations of Hermite and discovered by

Picard. Here the coefficients of the equation, in addition to being uniform functions of the variable, are doubly periodic functions having the same period, and the general integral is uniform. When this is the case the integral can be expressed in finite form by means of entire polynomials, the exponential function, and Jacobi's H-function.

Starting with these three categories of linear differential equations as a basis, Halphen, in his crowned "*Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables,*" proposed the following double problem:

(1) Having given a linear differential equation in the variable X and the unknown Y, to determine if there exists a substitution

$$x = \phi(X), \quad y = Y\psi(X),$$

such that, taking x for the new variable and y for the new unknown function, the transformed equation shall belong to one of the three categories:

I. Equations with constant coefficients;

II. Equations whose general integral is rational;

III. Equations whose general integral is uniform, and whose coefficients are doubly periodic functions having the same periods.

(2) Having found such a substitution, to effect the integration.

In solving a new problem the attempt is always made to simplify it by a series of suitable transformations; but there is a limit to these transformations, for in any problem there is, so to speak, something essential, which it is impossible for any transformation to alter. From this arises the importance of the general notion of invariants, which presents itself in every mathematical question. Laguerre was the first to introduce this conception of *invariant* into the theory of linear differential equations, which can thus be changed into their simplest possible forms. Brioschi has also made important contributions to the theory of the invariants of linear differential equations; but it was Halphen who, starting from his already established theory of differential invariants, has, in the memoir cited, made the most important contribution to the theory of these invariants, and employed it to solve the above problems. It is impossible to give here the results of Halphen's investigations, but two or three of his conclusions may be mentioned.

First, he says, in order that there may exist a substitution of the above form which will transform a given equation into one with constant coefficients, it is *necessary and sufficient* that its absolute invariants be constants.

Second, in order that there may exist a transformation of the above form which will transform a given equation into an

equation whose general integral is rational, or into an equation with doubly periodic coefficients whose general integral is uniform, *it is necessary but not sufficient* that the invariants be connected by algebraic relations of deficiency zero or one. For the further study of this second proposition Halphen's memoir must be consulted.

Two interesting results in the case of the second problem may also be mentioned here.

(1) Having given a differential equation in which the independent variable and the coefficients are rationally expressible in terms of a parameter α:

If the ratios of the integrals regarded as functions of α have only algebraical critical points in number three at most (infinity included), and, when these points are three in number, if their orders m, n, p satisfy the condition

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}>1;$$

we can make a change of the variable and the unknown function which will transform the equation into one whose general integral is rational.

(2) Having given a linear differential equation in which the variable and the coefficients are rationally expressible in terms of a parameter α:

If the ratios of the integrals considered as functions of α have only algebraical critical points (infinity included); and if these points are three in number and their orders satisfy the relation

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p}=1,$$

or if they are four in number and all of the second order; we can find a substitution which will change the equation into one whose coefficients are doubly periodic and whose general integral is uniform.

This last proposition is of particular interest as it gives new cases of integrability of Gauss's equation, an equation whose importance has been recognized by the ablest analysts, and upon which an enormous amount has been written. It is sufficient here to mention the names of Schwarz, Fuchs, Brioschi, and Klein. The new cases of integrability recognized by Halphen's theorem are when the equation can be integrated by aid of the elliptic functions. Halphen concludes the third chapter of his memoir, which deals with the general theory of invariants of linear equations of all orders, by indicating the method of procedure which it is desirable to

follow when in the study of the critical points we substitute the consideration of the invariants for the consideration of the equation itself. In closing his introductory chapter Halphen says, "*En résumé*, I have treated in this memoir a theoretical question relative to the transformation of linear equations into other equations belonging to those types for which we know the integration to be possible. The developments indispensable to the solution of this question constitute a part, less than half, of this work. All the rest is devoted to applications which I have thought it best to multiply, even at the risk of wearying the reader. But it seems to me that in such a matter as this applications treated completely ought to dominate, and that the aim to be pursued is this: to integrate effectively equations which we know to be integrable."

The notion of invariants arrived at by a substitution in the case of differential equations, is only part of a much more general theory which can only be briefly alluded to here. All branches of mathematics, whether pure or applied, are intimately connected the one with the other, and notions at first restricted to a special field of research are susceptible of receiving unforeseen extensions. Such is the notion of *group*, met with to-day in every branch of mathematical research. Poincaré in his "*Notice sur Halphen*" says: "The mode of procedure of mathematical science is always the same. It studies transformations of different natures; and, to that end, it must search for that which remains constant and unaltered during these transformations. Above all, it has for aim the study of groups and for means the search for invariants. This does not appear in every case with the same distinctness, but it is always true. If we can see at first sight that projective geometry is nothing else than the theory of linear substitutions, we do not perceive so quickly that elementary geometry is conducted to the theory of orthogonal substitutions."

Since the time of Galois the theory of groups of substitutions has played a rôle of the highest importance in algebra. An analytical theory presenting close analogy to Galois's theory has been developed by Lie in his "*Theorie der Transformationsgruppen*," a most admirable account of which by Dr. Chapman has recently appeared in the BULLETIN of this Society. Lie has made the discovery of prime importance, that the search for all these groups for a given number of variables and parameters is conducted to the integration of ordinary differential equations. Lie's theory is of the highest importance in the integral calculus, the real aim of which is to integrate differential equations; it is not confined to the transformations of points, but concerns itself with the

contact transformations, so important in the theory of partial differential equations. Lie has also considered the subject of continuous groups of infinite order and developed the general principles involved in the research for the invariants of differential equations.

The detailed investigations of Laguerre, Brioschi, and Halphen in the case of linear differential equations have already been mentioned. To these names must also be added that of Forsyth, who in his memoir "*Invariants, covariants, and quotient-derivatives associated with linear differential equations*" and in other papers has added important and interesting results to the general theory of the linear differential equations.

It must suffice here to merely mention a most interesting memoir by Appell on the study of the invariants and the cases of integrability:

(1) of equations of the form

$$\frac{dy}{dx} = \frac{a_0 + a_1 y + a_2 y^2 + \ldots + a_n y^n}{b_0 + b_1 y + \ldots + b_p y^p} \quad (p < n)$$

which preserve this form when we make the substitution

$$y = \eta u(x) + v(x), \quad \frac{d\xi}{dx} = \mu(x),$$

ξ being the new independent variable, and η the new unknown function;

(2) of algebraic differential equations which are homogeneous with respect to y and its derivatives and which preserve their form after the substitution

$$y = \eta u(x), \quad \frac{d\xi}{dx} = \mu(x).$$

The second part of this memoir is exceedingly interesting and suggestive, especially in its use and suggested generalization of a theorem of the author, by which, under certain conditions, the integration of an algebraic differential equation of any order and degree is conducted to the integration of a linear differential equation of an order one higher—an extension in a certain sense of the known property of Riccati's equation. The investigations of M. Roger Liouville on the invariants of non-linear differential equations of the first and second orders merit special attention, as also those of Rivereau and Painlevé, but they can only be referred to in this paper.

To speak of Poincaré's researches in differential equations is to speak of the most important discoveries in analysis of

modern times. His theory of fuchsian functions is connected at once with the theory of differential equations, since it enables us to integrate linear differential equations with algebraic coefficients, and with the general theory of functions, since these transcendents present certain remarkable peculiarities which are of such a nature as to cast considerable light on the manner of being of analytical functions. These fuchsian functions are generalizations of the modular functions, studied by Hermite in his researches on the elliptic functions, which possess an infinite number of singular points distributed along a circle. It will be sufficient to mention one single property, discovered by Poincaré, of these functions, for the geometer to recognize their importance. The fuchsian functions are of two kinds: one existing in the entire plane, the other existing only in the interior of the fundamental circle. In both cases there exists an algebraic relation between two fuchsian functions which have the same group. The determination of the deficiency of this relation is of the highest importance, and has been obtained by Poincaré both by analytical processes and by aid of the geometry of position. The existence of these relations makes it possible to utilize these functions in the study of algebraic functions and algebraic curves. Thus, *we can express the co-ordinates of a point of any algebraic curve whatever as fuchsian functions, that is as uniform functions of a single parameter.*

So profound a result is sufficient in itself to show the interest and importance attaching to these new functions; but, for the present purpose, it is desirable to give at least a faint idea of their relations to the linear differential equations. To attempt to give a full idea of the importance of these functions of Poincaré's, would be, for the present writer, to attempt the impossible; indeed, a mere statement of Poincaré's results produces an impression of exaggeration.

Since the time of Cauchy, mathematicians, recognizing the enormous difficulties and complications of the problem, have not attempted to study the nature of the integrals of differential equations, ordinary or partial, for all values of the variable,—that is, throughout the plane,—but have confined themselves to the investigation of the properties of these integrals in the neighborhood of certain given points. They thus perceived that these properties are very different according as we are concerned with an ordinary point or a singular point. The researches of Briot and Bouquet are too well known to need more than mention, and since their appearance most important additions to the theory of non-linear equations have been made, notably by Fuchs, Poincaré, Picard, and more recently by Painlevé. But the study of the integrals of differential equations in the neighborhood of a given point,

whatever may be its utility from the point of view of numerical calculation, can only be regarded as a first step in this study. These developments, which hold only in a very limited region, do not teach us concerning the integrals of differential equations that which the θ-functions teach us concerning the integrals of algebraic differentials : they cannot be considered as true integrations.

They must then be taken as a point of departure for a more profound study of the integrals of differential equations, in which it is proposed to free ourselves from the restrictions of these "limited regions," where, to use a phrase of Poincaré's, "*on s'était systématiquement cantonné*," and to follow the integral as the variable moves throughout the entire plane.

This generalized study can be entered upon from two points of view.

(1) We may propose to express the integrals by developments which *always* hold, and are no longer limited to a particular domain. This leads to the introduction of new transcendants into the theory ; but this introduction is necessary in any case, for the functions which have previously been introduced into analysis will only permit us to integrate a very small number of differential equations.

(2) This method of integration, though affording us a knowledge of the properties of the differential equations from the point of view of the theory of functions, is not in itself sufficient if, for example, we wish to apply the equations to questions of mechanics or physics. These developments would not readily teach us, for example, if the function was one which continually increased, if it oscillated between certain limits, or if it increased beyond all limit. In other words, if the function is considered as defining a plane curve we would not know the general form of the curve. In certain applications all these questions are of as much importance as the numerical calculation. We have here then a new problem for solution. These are the two problems that Poincaré proposed to himself and undertook and succeeded in the task of solving,—truly a task for a giant of intellect.

"Desiring," to quote Poincaré, "to express the integrals of differential equations by the aid of series *always* convergent, I was naturally led first to attack the linear differential equations. These equations, in fact, which have been during recent years the object of the investigations of Fuchs, Thomé, Frobenius, Schwarz, Klein, and Halphen, were the best known of all. We had for a long time possessed the developments of their integrals in the neighborhood of a given point, and in a quite large number of cases had succeeded in completely integrating them by aid of functions already known.

It was then in commencing my studies at this point that I had the most chances of arriving at a result.

"But it was further necessary to make some hypothesis concerning the coefficients of the equations which I wished to study. If, in fact, I had taken for coefficients *any functions whatever*, I would equally have obtained for integrals *any functions whatever*, and consequently would not have been able to say anything precise on the subject of the nature of these integrals, which was my aim. I was thus led to examine the linear equations with rational and with algebraic coefficients."

The following is Poincaré's classification of these linear equations, the one which, from the point of view of the proposed problem, is the most natural.

Let y be an integral of a linear equation of order n with rational coefficients. Write

$$z = e^{\int \lambda dx}\left(F_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + F_{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + F_1\frac{dy}{dx} + F_0 y\right),$$

where λ and the F's are rational functions of x. It is clear that z, like y, will satisfy a linear equation of order n with rational coefficients. Poincaré says that these equations belong to the same *family*. In fact, it is easily seen that the knowledge of the properties of the function y involves that of the properties of the function z. In each family there are an infinity of different equations, but certain functions of the coefficients have the same value for equations of the same family. In other words, there are *invariants* which remain unaltered by the above substitution. These invariants are not the same as those of Halphen for linear equations. These latter arise from a transformation which consists in replacing x by *any function whatever* of x' and multiplying y by *any other function whatever* of x'. On the contrary, the functions entering into Poincaré's substitution are not any whatever, but are rational. This shows well the difference between Halphen's investigations and those of Poincaré. Halphen sought above all to find the relations between different integrals, and could thus with impunity introduce any functions whatever into his calculations. Poincaré, on the other hand, sought to study the nature of the integral itself; this nature would manifestly be altered if the integral were multiplied by an arbitrary function, as is the case in Halphen's work.

This intimate study of the nature of the integrals can only be made by the introduction of new transcendents. These new transcendents have a close analogy to the elliptic functions, which is not surprising, as Poincaré was attempting to do for differential equations with algebraic coefficients what

had already been done by aid of the elliptic and Abelian theta-functions for the integration of algebraic differentials.

This analogy with the elliptic functions served Poincaré as a guide in all his researches. The elliptic functions are uniform functions which are unaltered when the variable is increased by certain periods. This notion is so useful in mathematical analysis that mathematicians had long seen the desirability of generalizing it by seeking uniform functions of a single variable x, which would remain unaltered when this variable was subjected to certain transformations. These transformations cannot be chosen arbitrarily, but must evidently form a group: and further, this group must contain no infinitesimal transformation; that is, x must not vary by an infinitesimal amount. For, if this were so, on repeating this transformation indefinitely, x would vary continuously; and our uniform function, remaining unaltered when the variable changed continuously, would necessarily reduce to a constant. In other words, the group of transformations must be discontinuous.

In elliptic functions the transformations of the group consist in adding constants. The functions are studied by dividing the plane into an infinite number of parallelograms of periods. All the parallelograms are obtained by transforming one of them by the different substitutions of the group; so that a knowledge of the function in one of these parallelograms involves a knowledge of it throughout the entire plane.

So if we consider a more complicated discontinuous group generated by a transcendent of higher order, we can divide the plane (or the part of the plane in which the function exists) into an infinity of regions, or curvilinear polygons, in such a way that we can obtain all these regions in applying to one of them the different transformations of the group. The knowledge of the function in the interior of one of these curvilinear polygons will involve the knowledge of the function for all possible values of the variable.

In the elliptic functions, considering the integrals of "the first kind," we regard the variable x by the process of inversion as a function of the integral. The function thus defined is uniform and doubly periodic. So, considering a linear equation of the second order, and by a species of inversion, Poincaré regards the variable x not as a function of an integral but as a function of the ratio of the two integrals of the equation. In certain cases the function thus defined is uniform, and then remains unchanged by an infinity of linear substitutions changing z into

$$\frac{\alpha z + \beta}{\gamma z + \delta}.$$

When this is the case, the group formed by these substitutions must be discontinuous, and the curvilinear polygons referred to must be limited by arcs of circles. Poincaré first supposes the coefficients of the substitutions

$$\left(z, \quad \frac{\alpha z + \beta}{\gamma z + \delta}\right)$$

to be real, or, what comes to the same thing, supposes that these substitutions do not alter a certain circle which he calls the "*fundamental circle.*" In this case the arcs of circles which serve as sides of the polygons cut the fundamental circle orthogonally.

What is then the condition that a group generated by a given curvilinear polygon shall be discontinuous? Poincaré finds this condition, and then constructs the discontinuous groups formed by substitutions which leave the fundamental circle unaltered, and which he calls fuchsian groups. Here an important problem now presents itself: having given a fuchsian group, do there exist functions which are unaltered by the substitutions of this group? In answering this question Poincaré appeals again to the analogy with the elliptic functions. These functions can be regarded as the quotients of two θ-series. These auxiliary transcendents are not only uniform, but are also entire functions; they are not doubly periodic, but reproduce themselves multiplied by an exponential when the variable is increased by a period. So in the new case the fuchsian functions are expressed as the quotient of two "*theta-fuchsian*" series, which are finite and uniform, perfectly analogous to the θ-functions, and reproduce themselves multiplied by a simple factor when we apply to the variable one of the substitutions of the group.

In order to complete the analogy with the elliptic functions, it was necessary that the other properties of these functions, such as addition, multiplication, and transformation, should be extended to the new transcendents. The theory of transformation is immediately generalized—always with this difference, that the group of fuchsian functions being very much more complicated than that of the elliptic functions, the cases to be considered are much more numerous and varied. A point of great interest in the extension of this theory of transformation is the new light thrown upon the reduction of the Abelian integrals. The theory of addition cannot be extended to all the fuchsian functions; it is only possible in one particular case and for one special class of these transcendents. The question need not, however, be entered into here.

In extending the linear substitutions to the case where the

coefficients are no longer restricted to be real, but are arbitrary, Poincaré arrives at the discontinuous groups which he calls kleinean, and so to a new class of functions, the kleinean functions, which are perfectly analogous to the fuchsian functions. The only difference which it is necessary to mention is that which arises from the form of the region inside of which these new functions exist. This region instead of being a circle is limited by a non-analytical curve which has no determinate radius of curvature; in other cases the region is limited by an infinity of circumferences. Another class of functions can only be mentioned. These are the zeta-fuchsian functions, which play the same part in the integration of the fuchsian functions that the zeta-functions play in the integration of the elliptic functions.

In concluding this very imperfect sketch of what Poincaré has done for the integration of linear differential equations, we can say that he has shown how to express the integrals of linear equations with algebraic coefficients by aid of these new transcendents in the same way as the integrals of algebraic differentials were previously expressed by aid of the Abelian functions. Further, these latter integrals are themselves susceptible of being obtained by aid of the fuchsian functions; and we thus have for them expressions entirely different from those given by the θ-functions of several variables.

Poincaré's further investigations on the integration of linear differential equations by aid of algebraic and Abelian functions, his work on non-linear equations (with one exception to be referred to later), on curves defined by differential equations, on irregular integrals, on partial differential equations, and on the differential equations of celestial mechanics, must be passed over. Enough has been said, however, to show the great value of his work. In leaving the subject of linear differential equations a mere mention will have to suffice of two interesting papers which have recently appeared : one by Appell, on linear differential equations transformable into themselves by change of the function and the variable; and one by Helge von Koch, on infinite determinants and linear differential equations.

The linear differential equations possess one remarkable property,—the singular points are the same for all the integrals In the case of equations whose coefficients are entire polynomials in x, these singular points are the values of x which annul the first coefficient. It is upon this circumstance that the method of integration of these equations by the zeta-fuchsian functions is founded. Non-linear equations do not in general possess this property. This has led to important in-

vestigations by Fuchs and Poincaré with the end of ascertaining whether or not there existed other classes of differential equations for which all the particular integrals had the same singular points. The researches of these mathematicians were confined to equations of the first order.

Fuchs found the necessary and sufficient conditions that an integral of the equation

$$f(x, y, y') = 0$$

where y, y' enter algebraically, shall have only fixed critical points x_i. These conditions once satisfied, if we wish the general integral to be uniform, it remains to express that these points x_i, which are known, are not critical points of the integrals. Poincaré resuming the question, arrived at conclusions as interesting as unexpected. He found that when the critical points of the equation

$$f(x, y, y') = 0$$

are fixed this equation can either be integrated algebraically, or by a quadrature, or can be conducted to a Riccati's equation.

Picard has undertaken the generalization of the work of Fuchs and Poincaré to the case of differential equations of the second order. He says: "It would seem at a first glance that the extension of the reasonings employed in the case of equations of the first order would be easy, but this is not the case. We may indeed commence by following the methods of reasoning employed by Poincaré; the end of his reasoning is unfortunately not applicable. We find ourselves always in presence of the same fact: A certain bi-uniform transformation is not necessarily bi-rational, and it is this fact which changes throughout the entire character of this theory." Picard confines himself to the case of bi-rational transformations. The difficulties in the way of a study of non-linear differential equations of the second order have caused this subject to remain as yet almost untouched; indeed, though in recent years much has been done, the non-linear equations of the first order still afford a vast field for research.

A problem of the highest importance connected with the differential equations of the first order and degree is to determine in any case whether or not the equation is integrable algebraically. To solve this problem it is manifestly sufficient to find a superior limit for the degree of the integral; after that the only operations remaining to be performed would be purely algebraical. This is a problem on the solution of which, it would seem, mathematicians might have been tempted to bestow much labor; nevertheless, they have concerned themselves very little with it. Indeed, from the time

of Darboux's masterly memoir in 1878, the problem was entirely neglected, until the Académie des Sciences, recognizing the importance of a more profound study of differential equations of the first order and degree, proposed as a subject of competition for the *Grand Prix des Sciences Mathématiques* for 1890 the following:

"To perfect in some important point the theory of the differential equations of the first order and first degree."

The outcome of the competition has been two important memoirs: one by M. Painlevé, to which was awarded the Grand Prize; and the other by M. Autonne, who received honorable mention.

M. Autonne takes for point of departure a geometrical interpretation of which every differential equation of the first order is susceptible. He shows that such an equation can be considered as giving curves situated on a certain algebraic surface, and of which the tangents belong to a certain linear complex. The surface is unicursal if the equation of the first order is at the same time of the first degree. Taking then a unicursal surface, the author forms the equation which corresponds to it, and which he calls "*réglementaire.*" His work then consists of a study of the equations of the first order and degree considered as "*réglementaires.*" Autonne makes a classification of the critical points of the equation. It will be sufficient to mention, in addition to the ordinary critical points forming the general case, those which he calls "*dicritical.*" These are points through which pass an infinite number of simple branches of integrals having an arbitrary tangent.

Painlevé's researches constitute one of the most important contributions ever made to the theory of the non-linear differential equations of the first order and degree. He considers any differential equation whatever of the first order where the function and its derivative figure algebraically, and makes at first an important distinction between the fixed and the movable critical points of the integrals,—by movable critical points meaning those which are susceptible of changes of position as the constant of integration changes. Such points cannot be points of indetermination for the integral. The utility of making this remark explicitly is seen if it is noticed that equations of an order higher than the first can have movable singular points. As an illustration, take the equation

$$(yy'' - y_2'')^2 + 4yy'^3 = 0,$$

whose general integral is

$$y = C_1 e^{\frac{1}{x - C_2}};$$

the essential singularities of the integrals depend evidently on the constant C_2.

Painlevé draws a system of cuts in the plane of the independent variable, which prevent this variable from turning round the fixed critical points, and then studies the equations for which the integrals take only a limited number of values around the movable critical points. Under this hypothesis we can conceive the general integral put into a form which will bring into evidence a class c of algebraic curves associated with the given equation.

Every curve of this class is a rational transform of the curve represented by the differential equation when for any fixed value whatever of the variable we regard the function and its derivative as the co-ordinates. If the deficiency of the curves c is greater than unity, we can by algebraic operations determine the class, and the integral itself can be obtained algebraically. If the curves c are of deficiency one, it may be necessary, in order to find the integral, to obtain a solution of a linear equation, and to ascertain if a certain Abelian integral has only two periods. Only the case of deficiency zero escapes this method; this circumstance necessarily presents itself when the equation is of the first degree with respect to the derivative.

Painlevé's methods can be applied in the attempt to ascertain whether or not the integrals of a given differential equation are algebraic, or can take only a limited number of values in the plane. The distinction between fixed and movable critical points permits the separation of this question into two parts, and in some cases the solution of this problem can be arrived at,—a solution which in all its generality will doubtless not soon be found.

The equalities and inequalities added by Autonne and Painlevé to those of Darboux constitute a very important progress in the solution of the problem of finding out whether or not a differential equation of the first order and degree is algebraically integrable; but, as Poincaré has pointed out in a recent paper in the *Rendiconti del Circolo Matematico di Palermo,* much remains to be done. Suppose, in fact, that the general integral is written

$$F = \text{const.},$$

where F is a rational function. Another form of the general integral will be given by equating to a constant any entire polynomial whatever in F. It follows as a consequence of this that the superior limit of the degree of the general algebraic integral cannot be found, at least unless some means

can be found of expressing in the inequalities that this integral is irreducible.

Painlevé himself clearly perceived this difficulty, but was unable to overcome it. He was unable to solve the problem in all its generality, but only to show that in a certain number of cases the integral could not be algebraic. As already stated, Painlevé's problem was to determine whether a given differential equation admitted an algebraic integral of a given deficiency. In the paper referred to, Poincaré gives a formula which contains the complete solution of Painlevé's problem whenever the dimension of the equation exceeds 4. He has also demonstrated some properties of equations which are integrable algebraically. Such results, he says, may not for the moment possess any great value, but they may acquire one the moment it is known whether these properties can be extended to non-integrable equations, or if they are not always true for these equations. In the first case we would have a general theorem applicable to all differential equations, and in the second case we would have a criterion which would enable us to demonstrate that the equations of certain categories were non-integrable.

In the subject of partial differential equations most important researches have been made during the past ten years, notably by Goursat, Poincaré, Picard, and Darboux. Indeed, the latter's treatise on the theory of surfaces might, from a certain point of view, be regarded as a treatise on these equations, particularly the equations of the second order. Picard has incorporated some of his more important results in the first fascicule of the second volume of his *Traité d'Analyse*. There also, among other methods for the proof of the existence of an integral of a differential equation, he gives his own elegant method of proof by successive approximations, an English translation of which by Dr. Fiske has been published in the BULLETIN of this Society. Picard has made some interesting extensions of this method, particularly to partial differential equations. The whole subject, however, of the recent developments of the theory of these equations and their applications to geometry must be reserved for another paper.

JOHNS HOPKINS UNIVERSITY,
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ON A GENERAL FORMULA FOR THE EXPANSION OF FUNCTIONS IN SERIES.*

BY PROF. W. H. ECHOLS.

1. If O be the symbol which represents any operation performed on a function, and O^r the repetition of that operation r times, then the formula referred to above is

$$\begin{vmatrix} fx & f_1x & \cdots & f_nx & f_{n+1}x \\ fy_1 & f_1y_1 & \cdots & f_ny_1 & f_{n+1}y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fy_p & f_1y_p & \cdots & f_ny_p & f_{n+1}y_p \\ Ofx_1 & Of_1x_1 & \cdots & Of_nx_1 & Of_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ O^qfx_q & O^qf_1x_q & \cdots & O^qf_nx_q & O^qf_{n+1}x_q \\ \Phi(u) & 0 & \cdots & 0 & 1 \end{vmatrix} \tag{1}$$

in which all elements of the last row except the first and last are zero. The symbol O^rfx_t means that after the rth operation on fx, the argument is changed into x_t. $\Phi(u)$ represents, in general, some function of $x, y_1, \ldots, y_p, x_1, \ldots, x_q$, involving also the form of the functions in the determinant.

If now the operation O be such that the Φ function may be so determined that the above determinant vanishes, we have, regarding x as the variable, the formulæ

$$\begin{aligned} fx &= A_1f_1x + \cdots + A_{n+1}f_{n+1}x, \\ fx &= B_1fy_1 + \cdots + B_pfy_p \\ &\quad + C_1Ofx_1 + \cdots + C_qO^qfx_q + D\Phi(u). \end{aligned}$$

The first of these may be regarded as an expansion of fx according to the functions $f_1x, \ldots f_{n+1}x$, whose coefficients are independent of the argument x, save in so far as Φ is a function of x. The second, in turn, may be regarded as an expansion of fx according to the form fy_r and the successive operatives of fx, whose coefficients are independent of the form of the function fx; the residual term being $D\Phi(u)$, wherein D does not depend on the form of the function fx.

* Read before the New York Mathematical Society, January 7, 1893. This paper is intended to be a brief exposition of the general theorem which is the basis of a series of papers entitled "On Certain Determinant Forms and their Applications," now in course of publication in the *Annals of Mathematics*.

We proceed to examine particular cases of (1) wherein the determinant may be either that of Interpolation, Differentiation, or Differences.

INTERPOLATION.

2. In (1), if $q = 0$, then each row, except the last, is obtained by specifying the variable in the first row, and we have, for the general formula of interpolation,

$$\begin{vmatrix} fx, & f_1x & \dots & f_nx, & f_{n+1}x \\ fx_1, & f_1x_1 & \dots & f_nx_1, & f_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fx_n, & f_1x_n & \dots & f_nx_n, & f_{n+1}x_n \\ \Phi(u), & 0 & \dots & 0, & 1 \end{vmatrix} = 0, \qquad (2)$$

wherein

$$\Phi(u) = \begin{vmatrix} f^nu, & f_1^nu & \dots & f_n^nu \\ fx_1, & f_1x_1 & \dots & f_nx_1 \\ \cdot & \cdot & \cdot & \cdot \\ fx_n, & f_1x_n & \dots & f_nx_n \end{vmatrix} \div \begin{vmatrix} f_1^nu & \dots & f_{n+1}^nu \\ f_1x_1 & \dots & f_{n+1}x_1 \\ \cdot & \cdot & \cdot \\ f_1x_n & \dots & f_{n+1}x_n \end{vmatrix},$$

in which f_r^nu means that f_rx is to be differentiated n times with respect to x and in the result x changed into u, which is some unknown quantity lying in value between the greatest and least values of the quantities $x, x_1, \dots, x_n$.

The proof follows:

Let M be the minor of $\Phi(u)$, and N be the minor of the element 1 in (2), and put

$$N = (-1)^n MR,$$

R being some unknown function of x.

Let us now assign to x some arbitrary constant value x_0, so that this equality becomes

$$N_0 = (-1)^n M_0 R_0,$$

which is independent of x.

Consider the function

$$F = N + (-1)^{n+1} MR_0.$$

This function vanishes when $x = x_0$ and also when x is equal to any one of the n quantities $x_1, \dots, x_n$. By Rolle's theorem, therefore, the first derivative of F must vanish for n values of x such as $u_1, \dots, u_n$, which lie respectively between the values $x_0x_1, x_1x_2, \dots, x_{n+1}x_n$. In like manner the

second derivative of F vanishes for $n-1$ values of x which lie respectively between the values u_1u_2, u_2u_3, . . . $u_{n-1}u_n$; and so on, until finally the nth derivative of F must vanish for some value, u, of x, which lies between the greatest and the least of the quantities $x_0, x_1, \ldots, x_n$, and we have

$$F_u^n = N_u^n + (-1)^{n+1}M_u^n R_0 = 0.$$

Since x_0 is arbitrary, we may drop the subscript, and write

$$R = (-1)^n N_u^n / M_u^n,$$
$$= \Phi(u).$$

Whence

$$N + (-1)^{n+1}\Phi(u)M = 0,$$

which demonstrates (2).

As we shall require, in the sequel, the result of the following, we proceed to give a particular illustration:

Let $f_r x = x^{r-1}$, then

$$\Phi(u) = \frac{f^n(u)}{n!}.$$

and we have

$$\begin{vmatrix} fx, & 1, & x & \ldots & x^{n-1}, & x^n \\ fa_1, & 1, & a_1 & \ldots & a_1^{n-1}, & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & a_n & \ldots & a_n^{n-1}, & a_n^n \\ \dfrac{f^n(u)}{n!}, & 0, & 0 & \ldots & 0, & 1 \end{vmatrix} = 0. \qquad (3)$$

Expanding this with respect to the first column, we obtain

$$fx = A_1 fa_1 + A_2 fa_2 + \ldots + A_n fa_n + A_{n+1}\frac{f^n(u)}{n!},$$

wherein

$$A_r = (-1)^{r+1}\frac{\zeta^{\frac{1}{2}}(x, a_1 \ldots a_{r-1}, a_{r+1} \ldots a_n)}{\zeta^{\frac{1}{2}}(a_1, \ldots, a_n)},$$

$$= \frac{(x-a_1)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n)}{(a_r-a_1)\ldots(a_r-a_{r-1})(a_r-a_{r+1})\ldots(a_r-a_n)}.$$

This is Lagrange's interpolation formula.

Let the values of the argument be equidistant with increment h, x being the greatest so that $x - a_r = rh$, then we have (E being the symbol of enlargement)

$$(-)_n h^n f^n(u) = fx - C_{n_1} E^r fx + \ldots + (-)^{r+1} C_{nr} E^r fx + \ldots + (-)^n E^n fx.$$

C_{nr} representing the binomial coefficient, and $E^r fx = f(x - rh)$. The member on the right of this equality is the well-known expression for the nth difference of fx, so we have

$$\Delta^n fx = h^n f^n(u), \tag{4}$$

wherein u lies between x and $x - nh$. If $n = 1$, then

$$f(x + h) - fx = hf'(u),$$

Lagrange's well-known form of Rolle's theorem. We may therefore consider (4) to be a generalization of this formula.

DIFFERENTIATION.

3. If in (1) the operation O be identical with the operation of Differentiation, we have for the corresponding general formula

$$\begin{vmatrix} fx, & f_1x & \ldots & f_nx, & f_{n+1}x \\ fy_1, & f_1y_1 & \ldots & f_ny_1, & f_{n+1}y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fy_p, & f_1y_p & \ldots & f_ny_p, & f_{n+1}y_p \\ f'x_1, & f_1'x_1 & \ldots & f_n'x_1, & f'_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^qx_q, & f_1^qx_q & \ldots & f_n^qx_q, & f^q_{n+1}x_q \\ \Phi(u), & 0 & \ldots & 0, & 1 \end{vmatrix} = 0. \tag{5}$$

In which, as before, u is an unknown value of x lying between the greatest and least of the quantities $x, y_1, \ldots, y_p, x_1, \ldots, x_q$. The bottom element of each column except the first and last is zero, and*

* In point of fact we should in the general form (5) write

$$\Phi(u) = \left(\frac{d}{dx}\right)^{n=p+q}_{x=u} \begin{vmatrix} fx & \ldots & f_nx \\ fy_1 & \ldots & f_ny_1 \\ \cdot & \cdot & \cdot \\ fy_p & \ldots & f_ny_p \\ f'x_1 & \ldots & f'_nx_1 \\ \cdot & \cdot & \cdot \\ f^qx_q & \ldots & f_n^qx_q \end{vmatrix} + \left(\frac{d}{dx}\right)^{n=p+q}_{x=u} \begin{vmatrix} f_1x & \ldots & f_{n+1}x \\ f_1y_1 & \ldots & f_{n+1}y_1 \\ \cdot & \cdot & \cdot \\ f_1y_p & \ldots & f_{n+1}y_p \\ f_1'x_1 & \ldots & f'_{n+1}x_1 \\ \cdot & \cdot & \cdot \\ f_1^qx_q & \ldots & f^q_{n+1}x_q \end{vmatrix},$$

because F vanishes $p + 1$ times for $x = x_0, y_1, \ldots, y_p$, therefore its

$$\Phi(u) = \begin{vmatrix} f^{q+1}u & \cdots & f_n^{q+1}u \\ fy_1 & \cdots & f_n y_1 \\ \cdot & \cdot & \cdot \\ fy_p & \cdots & f_n y_p \\ f'x_1 & \cdots & f_n' x_1 \\ \cdot & \cdot & \cdot \\ f^q x_q & \cdots & f_n^q x_q \end{vmatrix} \div \begin{vmatrix} f_1^{q+1}u & \cdots & f_{n+1}^{q+1}u \\ f_1 y_1 & \cdots & f_{n+1} y_1 \\ \cdot & \cdot & \cdot \\ f_1 y_p & \cdots & f_{n+1} y_p \\ f_1' x_1 & \cdots & f'_{n+1} x_1 \\ \cdot & \cdot & \cdot \\ f_1^q x_q & \cdots & f^q_{n+1} x_q \end{vmatrix}.$$

The proof follows:

Let M and N be the minors of $\Phi(u)$ and 1, respectively, in (5), and put

$$N = (-1)^n MR,$$

R being some unknown function of x. Assigning to x some arbitrary constant value x_0, we have

$$N_0 = (-1)^n M_0 R_0.$$

Consider the function

$$F = N + (-1)^{n+1} MR_0.$$

F vanishes when x takes any one of the values $x_0, y_1, \ldots, y_p$. Therefore, by the above, its derivative must vanish for some value of x, say u_0, which lies between the greatest and least of these values. This derivative vanishing also for $x = x_1$, then must the second derivative vanish for some value u_1, which lies between u_0 and x_1; which, in turn, vanishes again for $x = x_2$. Continuing thus, we find that the $(q+1)$th derivative of F vanishes for some value, u, of x lying between the limits prescribed above. Therefore

$$F_u^{q+1} = N_u^{q+1} + (-1)^{n+1} M_u^{q+1} R_0 = 0.$$

x_0 being arbitrary, we may strike off the subscript and put

$$\begin{aligned} R &= (-1)^n N_u^{q+1} / M_u^{q+1} \\ &= \Phi(u). \end{aligned}$$

Whence

$$N + (-1)^{n+1} \Phi(u) M = 0,$$

which is (5).

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first derivative vanishes p times between these values and also once more when $x = x_1$, and so on, until we find its qth derivative vanishing $p+1$ times among the values $x_0, y_1, \ldots, y_p, x_1, \ldots, x_q$, so that the $(q+p)$th or nth derivative must vanish once among them. The same thing would apply to the general formula for differences, etc.

The most interesting case of this general formula is when $p = 1$. It may then be written

$$\begin{vmatrix} fx, & 1, & \phi_1 x, & \ldots, & \phi_n x, & \phi_{n+1} x \\ fy, & 1, & \phi_1 y, & \ldots, & \phi_n y, & \phi_{n+1} y \\ f'a_1, & 0, & \phi_1' a_1, & \ldots, & \phi_n' a_1, & \phi'_{n+1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n, & 0, & \phi_1^n a_n, & \ldots, & \phi_n^n a_n, & \phi^n_{n+1} a_n \\ \Phi(u), & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \quad (6)$$

I have for want of a better name called the general form a composite, and the minor of fx is designated by the term *body-determinant*, or simply the *body* of the composite, in as much as it is got by striking out the outside rows and columns of the composite.

There are two classes of ϕ functions in (6) which require classification. The first class includes all of those functions which yield a body such that all elements on one side of its diagonal vanish, either through the operation of differentiation alone or through proper selection of the arbitrary constants. This class may be subdivided according as the elements above or below the diagonal vanish.* The second class includes those cases in which the elements on neither side of the body diagonal all vanish, the most interesting case of this class being that in which the body is a difference-product.

The first class yields readily all the well-known series, such as those of Taylor, Maclaurin, Bernouilli, Lagrange, Laplace, Abel, and a large number of other general series. The second class yields Fourier's theorem, and important general series in sines, cosines, Bessel's functions and logarithmic forms. A large number of these forms I have deduced in detail in the *Annals of Mathematics*, VI., 5; VII., 1, etc., with the object in view of illustrating the application of the composite to the deduction of special forms.

DIFFERENCES.

4. After demonstrating the general formula for interpolation we took notice of a special case for the purpose of deducing (4), the generalization of Lagrange's form of Rolle's theorem, because that theorem will now be needed for the establishment of the corresponding general formula for Finite Differences, which is, in the form corresponding to (6), as follows:

* The first division of this class is Wronski's expansion.

$$\begin{vmatrix} fx, & 1, & \phi_1 x \ldots & \phi_n x, & \phi_{n+1} x \\ fy, & 1, & \phi_1 y \ldots & \phi_n y, & \phi_{n+1} y \\ \Delta f x_1, & 0, & \Delta\phi_1 x_1 \ldots & \Delta\phi_n x_1, & \Delta\phi_{n+1} x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^n f x_n, & 0, & \Delta^n \phi_1 x_n \ldots & \Delta^n \phi_n x_n, & \Delta^n \phi_{n+1} x_n \\ \Phi(u), & 0, & 0 \ldots & 0, & 1 \end{vmatrix} = 0. \quad (7)$$

The proof follows:

Using M, N, and R in the same sense as before, we consider the function

$$F = N + (-1)^{n+1} M R_0.$$

$F = 0$ when $x = x_0$ and also when $x = y$; therefore its first derivative F' vanishes for some value of x which lies between x_0 and y, say u_0. Now when $x = x_1$, then $\Delta F = 0$; hence, if the scale of difference be h, in virtue of (4)

$$\Delta F = hF'(u)$$

(u lying between x and $x + h$), we have $F' = 0$ for some value of x between x_1 and $x_1 + h$, say $x_1 + h_1$. Since $F' = 0$ for u_0 and $x_1 + h_1$, then must $F'' = 0$ for some value of x, say u_1, between u_0 and $x_1 + h_1$.

Again, since by (5) we have

$$\Delta^2 Fx = h^2 F''(u)$$

(u between x and $x + 2h$), and since $\Delta^2 Fx = 0$ when $x = x_2$, then must $F'' = 0$ for some value of x between x_2 and $x_2 + 2h$, say $x_2 + h_2$. F'' vanishing for $x = u_1$ and $x = x_2 + h_2$, then must $F''' = 0$ for some value of x, say u_2, which lies between these values.

Reasoning in the same way, we proceed until finally we show that the $(n + 1)$th derivative of F must vanish for some value u, of x, which lies between the greatest and the least of the quantities x_0, y, $x_1 + h$, . . ., $x_n + nh$, so that we have

$$F_u^{n+1} = N_u^{n+1} + (-1)^{n+1} M_u^{n+1} R_0 = 0.$$

Dropping the suffix as before, we obtain

$$N + (-1)^{n+1} \Phi(u) M = 0,$$

which is (7).

Interesting forms of (7) are of course the general expansions in factorials, a number of which I have deduced, including as special cases the generalized forms of Taylor's and Maclaurin's series.

5. All expressions in the form of series which are deduced from the general composite are to be considered as having a finite number of terms and a terminal term R. These formulæ are not to be imagined as extending to infinity until it has been demonstrated that R becomes evanescent when n is infinite, and the coefficients in the series at the same time take on finite form. In general, it will be required of the functions in the composite, that they be finite, continuous, and single-valued between the values of the argument indicated, as also their successive operations. Under these circumstances when the member on the right converges to fx, on the left, as a limit when n becomes infinitely large, the result may be relied upon as arithmetically intelligible and true, the residual term R furnishing the evidence as to what values of the variable may be used in the equality. When we have determined the forms of the coefficients in the series, we have definitely determined the true analytical forms which these series must have if the expansion be possible, that is to say, we may regard the *qualitative* analysis as having been effected for these formulæ. The *quantitative* analysis remains yet to be done, that is the investigation of R, which determines the existence of the converging infinite series and the limits between which the variable can lie. The terminal term R is a function of an absolutely unknowable value of x, which can only be eliminated in the limit by showing that R vanishes when n is infinite.

The *rationale* illustrating the application of the composite to the expression of functions in infinite series may be presented thus :

Let there be two functions fx and

$$\sum_0^{n-1} A_r\phi_r x = A_0 + A_1\phi_1 x + \ldots + A_{n-1}\phi_{n-1}x.$$

Let the difference between these two functions be R, so that

$$fx = A_0 + A_1\phi_1 x + \ldots + A_{n-1}\phi_{n-1}x + R. \quad . \quad (8)$$

Let $a_1, \ldots, a_n$ be certain arbitrary values of the variable x, and let us have

$$\left.\begin{array}{l} fa_1 = A_0 + A_1\phi_1 a_1 + \ldots + A_{n-1}\phi_{n-1}a_1 + R_1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ fa_n = A_0 + A_1\phi_1 a_n + \ldots + A_{n-1}\phi_{n-1}a_n + R_n \end{array}\right\}. \quad (9)$$

In these n relations (9) there are n undetermined arbitrary quantities $A_0, A_1, \ldots, A_{n-1}$. Let us determine these so

that we shall have $R_1 = R_2 = \ldots = R_n = 0$. Thus, the value of A_r which satisfies this condition is,

$$A_r = (-1)^r \frac{\begin{vmatrix} fa_1, & 1, & \phi_1 a_1 & \ldots & \phi_{r-1} a_1, & \phi_{r+1} a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & \phi_1 a_n & \ldots & \phi_{r-1} a_n, & \phi_{r+1} a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}{\begin{vmatrix} 1, & \phi_1 a_1 & \ldots & \phi_{n-} a_1 \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}.$$

Consider the A coefficients to have these values. Taking now the $n+1$ relations (1) and (2), we have for the value of R,

$$\frac{\begin{vmatrix} fx, & 1, & \phi_1 x & \ldots & \phi_{n-1} x \\ fa_1, & 1, & \phi_1 a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_n, & 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}}{\begin{vmatrix} 1, & \phi_1 a_1 & \ldots & \phi_{n-1} a_1 \\ \cdot & \cdot & \cdot & \cdot \\ 1, & \phi_1 a_n & \ldots & \phi_{n-1} a_n \end{vmatrix}} = R.$$

We observe that the expansion of the determinant in the numerator of this ratio, according to its first row, gives the coefficient of $\phi_r x$ the value of A_r as determined above. We observe that this ratio for R takes the indeterminate form $0/0$, when the a's approach a limiting fixed value a. In order to evaluate the limiting value of this ratio as the a's approach the limit a, we apply to the numerator and denominator the operator

$$\left(\frac{d}{da_2}\right)'_{a_2 = a} \ldots \left(\frac{d}{da_n}\right)^{n-1}_{a_n = a},$$

obtaining

$$\frac{\begin{vmatrix} fx & 1 & \phi_1 x & \ldots & \phi_{n-1} x \\ fa & 1 & \phi_1 a & \ldots & \phi_{n-1} a \\ f'a & 0 & \phi_1' a & \ldots & \phi'_{n-1} a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1} a & 0 & \phi_1^{n-1} a & \ldots & \phi^{n-1}_{n-1} a \end{vmatrix}}{\begin{vmatrix} \phi_1' a & \ldots & \phi'_{n-1} a \\ \cdot & \cdot & \cdot \\ \phi_1^{n-1} a & \ldots & \phi^{n-1}_{n-1} a \end{vmatrix}} = R. \quad (10)$$

It is to be distinctly observed that in this process we do not require the functions fx and $\sum_0^{n-1} A_r\phi_r x$ to have a contact of the $(n-1)$th order at $x = a$ in order that we may equate their first $n-1$ derivatives when $x = a$. What we require is merely that the functions fx and $\varphi_r x$ $(r = 1 \ldots n-1)$ shall each have a determinate derivative at $x = a$, up to the $(n-1)$th operation. Of course, if fx and $\sum_0^{n-1} A_r\phi_r x$ have an $(n-1)$th contact at $x = a$, then our value for R holds true as well; but it is not dependent on such a relation: it simply includes it.

If now the successive functions $\phi_r x$ $(r = 1 \ldots n)$ may be formed in succession indefinitely according to a given law so that we may make r in $\phi_r x$ as great as we choose, then if it can be shown that R has for its limit zero, as r becomes infinite and at the same time the A's have limiting values such that $\sum^{\infty} A_r\phi_r x$ is a converging series, then we may write

$$fx = A_0 + A_1\phi_1 x + A_2\phi_2 x + \ldots \text{ ad. inf.}$$

The value of R has been shown to be

$$\frac{\begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix} \left(\frac{d}{dx}\right)^n_{x=u} \begin{vmatrix} fx, & 1, & \phi_1 x & \ldots & \phi_{n-1} x \\ fa, & 1, & \phi_1 a & \ldots & \phi_{n-1} a \\ f'a, & 0, & \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0, & \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix}}{\begin{vmatrix} \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot \\ \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix} \left(\frac{d}{dx}\right)^n_{x=u} \begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix}}, \quad (11)$$

in which u is some unknown value of x lying between x and a.

ON THE EARLY HISTORY OF THE NON-EUCLIDIAN GEOMETRY.

BY EMORY McCLINTOCK, LL.D.

It has until recently been supposed that the earliest work on non-euclidian geometry was Lobatschewsky's.* A much earlier production (1733) has been brought into notice by

* See BULLETIN of November, 1892, vol. II, No. 2, "On the Non-Euclidian Geometry."

Beltrami,* the author of which was Geronimo Saccheri, a Jesuit father of Milan, where he had charge of the Collegio di Brera. Under the title "Euclides ab omni Nævo vindicatus," this writer produced a work which, when we recall the many later futile discussions of the parallel-axiom, must be counted as marvellous. It is to be hoped that the society of which he was an ornament, and to which the many scientific achievements of its members are a just cause for pride, may reproduce for the benefit of future readers a treatise which may otherwise be known only at second-hand.

With our present knowledge, which leads to distinctions of distance-measurement, we may designate that (euclidian) geometry in which the fourth side of an attempted rectangle is equal to the side opposite as the geometry of equal distance; that of Lobatschewsky, in which it is greater, as the geometry of greater distance; and that third system, in which it is less, as the geometry of smaller distance. Saccheri dealt with the angles adjacent to the fourth side, deriving thence three possible systems, which he named respectively "hypothesis anguli recti," "hypothesis anguli acuti," and "hypothesis anguli obtusi," the first, of course, being euclidian, and proving as to each system "si vel in uno casu sit vera, semper in omni casu illa sola est vera." Other propositions were successively established, as for instance, "Ex quolibet triangulo, cujus tres simul anguli æquales sint, aut majores, aut minores, duobus rectis, stabilitur respective hypothesis aut anguli recti, aut anguli obtusi, aut anguli acuti," and "Esto quodvis triangulum AHD rectangulum in H. Tum in AD continuata sumatur portio DC æqualis ipsi AD, demittaturque ad AH productam perpendicularis CB: dico stabilitum hinc iri hypothesim aut anguli recti aut anguli obtusi aut anguli acuti, prout portio HB æqualis fuerit aut major aut minor ipsa AH."

Saccheri proved the "hypothesis anguli obtusi" untrue, as being incompatible with customary axioms; but the "hypothesis anguli acuti" caused him much greater difficulty, just as it subsequently did to Legendre. He confessed to a distracting heretical tendency on his part in favor of the "hypothesis anguli acuti," a tendency against which, however, he kept up a perpetual struggle (diuturnum prœlium). After yielding so far as to work out an accurate theory anticipating Lobatschewsky's doctrine of the parallel-angle, he appears to have conquered the internal enemy abruptly, since, to the surprise of his commentator Beltrami, he proceeded to announce dogmatically that the specious "hypothesis anguli acuti" is positively false.

* "Un Precursore Italiano di Legendre e di Lobatschewsky": *Rendiconti, R. Accad. dei Lincei*, 1889, I, 441.

The relation of Gauss to the work of the other non-euclidian pioneers, Lobatschewsky, Bolyai, Riemann, is so obscure as to constitute a historical problem.* Jacobi, writing to Legendre in 1828 on another subject, accuses Gauss of spreading a veil of mystery over his work. On this subject Gauss himself wrote to Bessel in 1829 that he had busied himself with it for nearly forty years, had made very extensive researches, and meant to publish nothing. In 1831 he wrote to Schumacher giving various details of what he called the non-euclidian geometry, including the important formula for the circumference of a circle published later by the younger Bolyai. In his youth he had discussed the subject actively with the elder Bolyai. Another, and according to Houël the close, friend of Gauss's youth was Bartels, who in 1807 went to Kazan, where subsequently Lobatschewsky became his chief pupil, his assistant, and his successor. When Bolyai's work appeared, Gauss wrote approvingly to his father, implying that the results agreed with his own, which would not be published; and when Lobatschewsky's treatise in German came out in 1840, Gauss expressed his hearty approval, though saying he had already obtained substantially the same theory. A little later he secured for Lobatschewsky an election to the Royal Society of Göttingen. There is no question as to Lobatschewsky's priority of publication, beginning with a lecture at Kazan in 1826 followed by repeated essays in Russian, or as to the originality of his work as a whole, whatever suggestions may have come from Bartels at the beginning. And there is no proof that whatever is common between Gauss and the Bolyais was uniformly original with Gauss. He was, however, the great genius who was for many years intent upon the subject, and between whom and all others interested in it during his lifetime there are plain lines of connection. His illustrious pupil Riemann took a different point of view, starting with Gauss's formula for the measure of curvature, and must have had at least some slight acquaintance with the Gauss-Lobatschewsky trigonometry valid for his suggested sur-

* In the paper already cited I followed Beez in stating too strongly the probable connection between Gauss and Lobatschewsky. I am indebted for my first knowledge of Beltrami's account of Saccheri to a letter from Professor Beez, in which he admits his mention of Gauss as the teacher of Lobatschewsky to be partly inferential, and not to be taken literally. Klein (Lectures, published 1892) says of Gauss: "Aber es ist auch keinem Zweifel unterworfen, dass er durch seinen Einfluss die Unter-Suchungen von Lobatscheffsky und Bolyai angeregt hat. . . . Von Bartels wurde Lobatscheffsky ganz ausführlich mit den Gaussischen Schriften bekannt gemacht, und es kann kaum bezweifelt werden, dass er von diesem auch eingeweiht wurde in die Fragestellung der Nicht-Euklidischen Geometrie."

faces of negative curvature,* although even of this, were any one to dispute it, there is probably no extant evidence which would be available in a court of law.

MORRISTOWN, *February* 18, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, February 4, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Professor Heinrich Maschke, University of Chicago; Lieutenant C. De Witt Willcox, U. S. A., U. S. Military Academy, West Point; Mr. J. N. James, U. S. Naval Observatory, Washington; Mr. Abraham Cohen, Johns Hopkins University. The council announced the adoption of the following resolution: "That any member of the society in good standing who is connected with an educational institution may order one extra copy of the BULLETIN for the use of such institution at the price of $2.50 a year."

Professor Thomas Craig read a paper entitled "Some of the developments in the theory of ordinary differential equations between 1878 and 1893." This paper appears in the present number of the BULLETIN, p. 119.

Dr. McClintock mentioned his having recently devised the following continued products, wherein $y = x - x^2$:

$$\frac{\sin(x\pi)}{\pi} = y(1+y)\left(1+\frac{y^2}{2^2[1+y]}\right)\left(1+\frac{y^2}{3^2[2+y]}\right)\cdots\left(1+\frac{y^2}{r^2[r-1+y]}\right)\cdots$$

$$= y\frac{2+y}{2-y}\left(1-\frac{y^2}{2^2[3-y]}\right)\left(1-\frac{y^2}{3^2[4-y]}\right)\cdots\left(1-\frac{y^2}{r^2[r+1-y]}\right)\cdots$$

$$= x\frac{1-x^2}{1+x^2}\left(1+\frac{x^2-x^4}{1.2[2+x^2]}\right)\left(1+\frac{x^2-x^4}{2\ 3[3+x^2]}\right)\cdots\left(1+\frac{x^2-x^4}{r[r-1][r+x^2]}\right)\cdots$$

If $x = \frac{1}{2}$, these become

$$\frac{1}{\pi} = \frac{5}{16}\left(1+\frac{1}{4^2.5}\right)\left(1+\frac{1}{6^2.9}\right)\cdots = \frac{5}{16}\left(1+\frac{1}{80}\right)\left(1+\frac{1}{324}\right)\cdots$$

$$= \frac{1}{4}\cdot\frac{1^2.5}{2^2.1}\cdot\frac{3^2.9}{4^2.5}\cdot\frac{5^2.13}{6^2.9}\cdots = \frac{9}{28}\left(1-\frac{1}{4^2.11}\right)\left(1-\frac{1}{6^2.15}\right)\cdots$$

$$= \frac{9}{28}\left(1-\frac{1}{176}\right)\left(1-\frac{1}{540}\right)\cdots = \frac{1^2}{8}\cdot\frac{3^2.3}{2^2.7}\cdot\frac{5^2.7}{4^2.11}\cdots$$

$$= \frac{3}{10}\left(1+\frac{1}{24}\right)\left(1+\frac{1}{104}\right)\left(1+\frac{1}{272}\right)\left(1+\frac{1}{560}\right)\left(1+\frac{1}{1000}\right)\cdots$$

* Riemann hat freilich schon 1854 die Beziehung sehr wohl gekannt." —Klein, Vorlesung, i. 191.

Proof as to each is had by dividing each factor by the corresponding factor of the well-known expression

$$\frac{\sin(x\pi)}{\pi} = x(1-x^2)\left(1-\frac{x^2}{2^2}\right)\cdots\left(1-\frac{x^2}{r^2}\right)\cdots,$$

the quotient in each case forming a continued product readily proved equal to unity.

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[1] Entered by mistake under the name Park, instead of Benjamin, in the last number of the Bulletin, p. 116.

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	P. M.		P. M.		P. M.
Saturday, Jan. [illegible]	[illegible]	Saturday, April 1	[illegible]	Saturday, [illegible]	[illegible]
Feb. [illegible]		May [illegible]		Nov. [illegible]	
Mar. [illegible]		June [illegible]		Dec. [illegible]	

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OF MATHEMATICAL SCIENCE.

EDITED BY

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THE ELLIPTIC FUNCTIONS.

The Applications of Elliptic Functions. By ALFRED GEORGE GREENHILL, M.A., F.R.S., Professor of Mathematics in the Artillery College, Woolwich. London and New York, Macmillan & Co., 1892. 8vo, pp. xi + 357.

IN recent years there has been a tendency to regard pure and applied mathematics as subjects to be kept clearly distinct. The increasing necessity for close specialization, occasioned by the enormous yearly output of memoir literature, is no doubt largely responsible for this unfortunate condition of things, but there are other elements which enter into the question. To a large class of pure mathematicians the lack of logical rigor in the current proofs of physical theorems is abhorrent; to them it seems that a treatise on any subject in pure mathematics should use methods peculiar to that subject, and that no appeal should be made to extraneous physical considerations. The applied mathematician, on the other hand, is too apt to speak scornfully of the works of his rival. It is indisputable that the problems which confront the physicist of to-day require for their adequate treatment the most refined methods in pure analysis. It is, we believe, equally certain that many portions of modern mathematics must remain, so to speak, in the air, until they are connected with the physical considerations from which they originated. Professor Greenhill's sympathies are largely with the practical man as against the theorist; the first words of his introduction are a quotation from Fourier in which occur the words "les questions vagues et les calculs sans issue." At the same time he has earned for himself a high reputation by his researches on the complex multiplication of elliptic functions, a subject without any very immediate applications. Professor Greenhill has rendered an important service to the mathematical public by showing how easily elliptic functions lend themselves to the solution of dynamical and other problems; an especially valuable feature of his treatise is the free use which he has made of the Weierstrassian functions.

In the first chapter there is a detailed treatment of pendulum motion, and in the course of this discussion several of the fundamental properties of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ are established. It is shown that they have a real period and that they may degenerate into circular and hyperbolic functions. There are obvious advantages in beginning the subject with a concrete example; but we doubt whether this is the way "best calculated to define the elliptic functions, and to give the student an idea of their nature and importance." The all-important

property of an elliptic function is its double periodicity, but all discussion of this is postponed till page 254.

In the second chapter there is an exhaustive account of the numerous forms which can be taken by elliptic integrals, and free use is made of the inverse notation sn^{-1}, cn^{-1}, dn^{-1}, $\wp^{-1}$. In the next chapter there are many geometrical and mechanical illustrations of the elliptic functions. Without entering into details we shall show how some of these problems lead to elliptic integrals.

(1) The equation for a central orbit described under a force P to the pole is

$$P = h^2u^2(u + d^2u/d\theta^2).$$

Hence, when $P = \mu u^3 + \nu u^5$,

$$d\theta = du/\sqrt{a + bu^2 + cu^4}.$$

(2) Let AB be an elastic rod with two rigid arms Aa, Bb at its extremities; and let two equal and opposite forces T be applied at a, b so as to act inwards. The rod being straight when unstrained, what is its form when bent? (See Minchin's "Statics," vol. II., p. 204).

Let a, b lie in the axis of x. For the equilibrium of the portion aAP, where P is a point (x, y) of the rod, it is necessary that the moment of T about P be equal to A/ρ, where A is the flexural rigidity and ρ the radius of curvature. The resulting equation $Ty = \rho$ leads to $Ad^2\psi/ds^2 = Tdy/ds = -T\sin\psi$; an equation of the same type as that which arises in pendulum motion.

(3) Captain Sumner's method determines the position of a ship at sea by the intersections of two small circles, known as circles of position. The projection of a circle of position upon a Mercator chart has for its equation

$$\text{either} \quad \cosh y/c = \sec\alpha \cos x/c,$$
$$\text{or} \quad \sinh y/c = \tan\beta \cos x/c.$$

In either case ds/dx leads to an elliptic integral.

(4) Let a surface of revolution be formed by the revolution of a plane curve round the axis Ox, and let a geodesic on this surface be projected upon a plane perpendicular to this axis. Let r, θ be the polar coördinates of a point on this projection. If the surface is formed by the revolution of a catenary, we have the equations

$$r = a\cosh x/a,$$

$$r^2d\theta/ds = h,$$

$$ds^2 = dx^2 + r^2d\theta^2 + dr^2.$$

These lead to

$$d\theta = b\,dr/\sqrt{(r^2 - a^2)(r^2 - b^2)}.$$

In the fourth chapter the addition theorems are proved for Jacobi's functions, and there is an interesting account of the poristic theorems connected with Poncelet's polygons. Also many applications are given of elliptic functions to spherical trigonometry.

The subjects of the remaining chapters (v. to x.) are the algebraical form of the addition theorem; the elliptic integrals of the second and third kind; the elliptic integrals in general and their applications; the double periodicity of the elliptic functions; the resolution of the elliptic functions into factors and series; and the transformation of elliptic functions. In these chapters the chief properties are stated in Weierstrass's as well as in Jacobi's notation, and in connection with these properties there are many illustrations. For example, the chief formulæ are used, with much skill, in the discussion of the revolving chain, the tortuous elastica, the spherical pendulum and the top, and the trajectory of a projectile for the cubic law of resistance. Particular mention should be made of the interesting treatment of the motion of a rigid body about a fixed point, when acted on by no forces, and of the accompanying account of various geometric constructions for the motion, including Sylvester's method for measuring the time (see Routh's "Rigid Dynamics," vol. II.).

As a geometric illustration of the importance of Weierstrass's formulæ in $\wp$- and σ-functions, we will show that some of the leading properties of the Cartesian can be derived from standard formulæ in Schwarz's "Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen." *

Let

$$z = \int_{\infty}^{w} dw/\sqrt{4w^3 - g_2 w - g_3}, \quad \text{or} \quad w = \wp(z),$$

where g_2, g_3 are real, and $g_2^3 - 27g_3^2$ is positive. Let the real roots e_1, e_2, e_3 of the equation $4x^3 - g_2 x - g_3 = 0$ be in descending order of magnitude. The parallelogram of periods is a rectangle formed by a real period $2\omega_1$ and by a purely imaginary period $2\omega_3$. Let 0, a, $\omega_1 - a$, ω_1, $\omega_1 + a$, $2\omega_1 - a$, $2\omega_1$ be points on the side (0 to $2\omega_1$), and let lines 1,

* The method employed in the text agrees essentially with that of Prof. Greenhill; the relations for the focal distances of a point are simplified by the use of the triple focus, and thus assume a more final form. The formulæ in the text have been published by Prof. Morley in the Haverford College Studies.

2, 3, 4 be drawn through a, $\omega_1 - a$, $\omega_1 + a$, $2\omega_1 - a$ parallel to the axis of imaginaries to terminate in the opposite side of the parallelogram of periods. When z describes the line 1 we have, in circular coördinates,

$$z + \bar{z} = 2a.$$

In the plane of $\wp z$ (or shortly the $\wp$-plane) the circular coördinates are $\wp z$, $\wp \bar{z}$ (or shortly $\wp$, $\bar{\wp}$). Let c denote $\wp 2a$. By the addition theorem, as in Halphen, vol. II. p. 335,

$$[\wp\bar{\wp} + c(\wp + \bar{\wp}) + g_2/4]^2 = 4(\wp + \bar{\wp} + c)(c\wp\bar{\wp} - g_3/4).$$

The equation is that of a Cartesian, since the highest power of $\wp$ is multiplied by $(\bar{\wp} - c)^2$. The triple focus has for its coördinates c, c, and corresponds to the point $z = 2a$. The single foci are at e_1, e_2, e_3. The curve depends on a single constant c or $\wp 2a$. It therefore includes the representations of four lines in the rectangle, viz. 1, 2, 3, 4. For when z is at any one of the points a, $\omega_1 - a$, $\omega_1 + a$, $2\omega_1 - a$, the values of $\wp 2z$ and $\wp 2a$ are the same. The lines 1 and 4 give the same values of $\wp z$ and give the outer oval of the Cartesian; while the lines 2 and 3 give the inner oval.

Let each point $\wp z$ be denoted (to periods *près*) by its parameter z, with the exception of the three points e_1, e_2, e_3. The point e_3 lies on the real axis outside the outer oval and e_2, e_1 lie within the inner oval. Starting from e_3, the real axis meets the curve in the points $\omega_2 + a$, $\omega_2 + a$, $\omega_1 + a$, a, where $\omega_2 = -(\omega_1 + \omega_3)$. For the sake of symmetry let us denote a point on the curve by the parameter $a + z$, in preference to z; then for points on the outer oval z is a pure imaginary, and for points on the inner oval $z - \omega_1$ is a pure imaginary (periods not counting). Since

$$\{\wp(z + \omega_\lambda) - e_\lambda\}\{\wp z - e_\lambda\} = (e_\mu - e_\lambda)(e_\nu - e_\lambda),$$

it follows that the point $a + z + \omega_\lambda$ is derived from the point $a + z$ by quasi-inversion (i.e., ordinary inversion plus reflexion in the real axis). Hence the point $a + z + \omega_\lambda$ is the inverse of the point $a - z$. The curve is therefore *its own inverse with regard to any focus.*

In formulæ (3), p. 47 of Schwarz's "Formeln und Lehrsätze," put

$$a = 0, \qquad b = 2a, \qquad c = a + z, \qquad d = a - z.$$

Therefore

$$a' = 2a, \qquad b' = 0, \qquad c' = z - a, \quad d' = a + z,$$
$$a'' = a + z, \quad b'' = a - z, \quad c'' = 0, \qquad d'' = -2a.$$

The formula (5) p. 48, namely,

$$(e_\mu - e_\nu)\sigma_\lambda a\ \sigma_\lambda b\ \sigma_\lambda c\ \sigma_\lambda d + (e_\nu - e_\lambda)\sigma_\mu a'\ \sigma_\mu b'\ \sigma_\mu c'\sigma_\mu d',$$
$$+ (e_\lambda - e_\mu)\sigma_\nu a''\sigma_\nu b''\sigma_\nu c''\sigma_\nu d'' = 0,$$

becomes

$$\Sigma(e_\mu - e_\nu)\sigma_\lambda 2a\ \sigma_\lambda(a + z)\ \sigma_\lambda(a - z) = 0.$$

But

$$\frac{\sigma_\lambda(a + z)\sigma_\lambda(a - z)}{\sigma(a + z)\sigma(a - z)} = \sqrt{\{\wp(a + z) - e_\lambda\}\{\wp(a - z) - e_\lambda\}} = \rho_\lambda.$$

where ρ_λ is the focal distance; and, similarly,

$$\frac{\sigma_\lambda 2a}{\sigma 2a} = \sqrt{\alpha_\lambda},$$

where α_λ is the distance from the triple focus to a single focus. Hence Schwarz's formula gives, after division by $\sigma 2a\ \sigma(a + z)\sigma(a - z)$,

$$\Sigma(e_\mu - e_\nu)\sqrt{\alpha_\lambda}\,\rho_\lambda = 0;$$

or, since

$$e_\mu - e_\nu = -(\alpha_\mu - \alpha_\nu),$$

$$\Sigma(\alpha_\mu - \alpha_\nu)\sqrt{\alpha_\lambda}\,\rho_\lambda = 0.$$

It can be proved, in a similar manner, that

$$\sqrt{\alpha_2\alpha_3}\,\rho_1 - \alpha_2\alpha_3 = \sqrt{\alpha_3\alpha_1}\,\rho_2 - \alpha_3\alpha_2 = \sqrt{\alpha_1\alpha_2}\,\rho_3 - \alpha_1\alpha_2$$
$$= \tfrac{1}{2}(\rho^2 - \alpha_2\alpha_3 - \alpha_3\alpha_1 - \alpha_1\alpha_2),$$

where ρ is the distance of the point from the triple focus.

When the discriminant $g_2^3 - 27g_3^2$ is negative, the periods $2\omega_1$, $2\omega_3$ are conjugate. Also if e_3 be the real root, e_1 and e_2 are conjugate. Let

$$\omega_2 = \omega_1 + \omega_3,$$
$$\omega_2' = \omega_1 - \omega_3.$$

Then ω_2 is on the real axis, ω_2' on the imaginary axis. The curve which corresponds in the $\wp$-plane to $z + \bar{z} = 2a$ is still

a Cartesian, but there is now only one oval. One real focus and two imaginary foci are collinear. The property of inversion now belongs only to the focus e_2, and of the relations between the focal distances the only one which is free from imaginaries is

$$\rho^2 = 2\sqrt{\alpha_2\alpha_1}\,\rho_2 + \alpha_1\alpha_2 + \alpha_2\alpha_3 - \alpha_1\alpha_3,$$

ρ, ρ_2, α_2 having their former meaning, and α_1, α_3 being the conjugate quantities $\wp 2a - e_1$ and $\wp 2a - e_3$.

The author states expressly in his preface that, as his book is addressed to trained mathematicians, he has considered it sufficient in many places to state the results without introducing the intermediate analysis. His opinion is that "by developing the analysis as it is required for some particular problem in hand, the student of applied mathematics will obtain a working knowledge of the subject of elliptic functions, such as he would probably never acquire from a study of a treatise like Jacobi's "Fundamenta Nova." This hand-to-mouth plan of taking up the subject has been carried, we think, too far. It would be a piece of pedantry to require that much space should be given to convergency considerations; but we are decidedly of opinion that such a proof as that on p. 202 ought not to have been inserted. It will certainly not bear close examination. It is unfortunate that so able a writer should show uniformly a contempt for processes and attach an undue importance to results; but much may be forgiven to one who can handle his symbols with such consummate ease. Wherever we open the book, we are struck with the freshness and originality of the treatment; though at times Mr. Greenhill's unconventionality is a trifle provoking. Was it necessary to raise the vexed question of pound-weight versus pound-mass on the first page, or to end the last chapter with the word, etc.?

We must call attention to one or two errors which have crept into the text. Chapter v. commences with the general symmetric quadri-quadric function of x, y. Two symmetric forms are to be distinguished:

(i.) $\alpha x^2y^2 + 2\beta xy(x+y) + \gamma(x^2+4xy+y^2) + 2\delta(x+y) + \epsilon = 0$,

which states that y is one of the polar pair of x with regard to a fixed quartic

$$\alpha t^4 + 4\beta t^3 + 6\gamma t^2 + 4\delta t + \epsilon = 0,$$

and

(ii.) the same expression $+ 4\mu(x-y)^2 = 0$,

which is the general symmetric quadri-quadric relation. The second form is that to which any quadri-quadric relation can be brought by writing $x = (lx' + m)/(l'x' + m')$. This may perhaps be inferred from the fact that the relation (ii.) contains five effective constants, while the general relation has eight, the ratios l/m', m/m', l'/m' accounting for the difference. (For a formal proof see Halphen's "Fonctions Elliptiques," vol. II. p. 338). Prof. Greenhill falls into two errors. He takes the special form (i.) as equivalent to

$$(\sqrt{X} - \sqrt{Y})^2/(x - y)^2 = a(x + y)^2 + 4b(x + y) + C,$$

where he should take the general form (ii.), and then, to compensate, he regards $(lx' + m)/(l'x' + m')$ as a function with four effective constants. On page 313 the incorrect statement is made that the elliptic functions of the nth parts of the periods are called by Klein *modular functions*. A sentence on page 175 is so worded as to convey to the reader the impression that, in the most general case, abelian functions are the one-valued functions of p arguments which are met with when we consider the integrals

$$\int (1, x, x^2, \ldots, x^{p-1})\, dx/\sqrt{X},$$

where X is a rational integral function of x of the degree $2p + 2$. These are obviously mere oversights on the part of the author. We have pointed them out because they may cause trouble to the beginner.

We believe that the many sterling merits of this treatise will cause it to pass rapidly through a first edition. We hope that Prof. Greenhill will be encouraged, by the success of this book, to write one on the applications of automorphic functions. A sentence in his recent address before the London Mathematical Society indicates that he is interested in this subject.

J. HARKNESS.

BRYN MAWR, *March* 9, 1893.

A CASE OF NON-EUCLIDIAN GEOMETRY.*

BY PROF. W. WOOLSEY JOHNSON.

THE note here presented was suggested by the very interesting article by Dr. McClintock in the November number (vol. II., p. 21) of the BULLETIN, and merely presents in another point of view the pseudo-measurements, occurring in the Cayley interpretation in the case in which c is positive, developed by Dr. McClintock (page 27 *et seq.*) under the name of *projective distances* as distinguished from the real distance of Euclidian geometry.

In the theory of projective metrics, the case is the symmetric one of the opposite variety from that of Lobatschewsky, or that in which the absolute quadric of Cayley is an imaginary sphere.

In the explanation given in the BULLETIN it will be remembered that, for the geometry of the plane, we have only to assume a *central point* of the plane, and a sphere of fixed radius touching it at that point; then for any finite straight line or *sect* in the plane (not simply a sect of a line passing through the central point) the *projective measure* is the length of the arc of a great circle which is the central projection of the sect upon the spherical surface. In like manner the *projective measure* of an angle is the spherical angle between the great circles which are the projections upon the sphere of the sides of the given angle. Thus a plane triangle is represented by a certain spherical triangle, and we develop a non-euclidian geometry of two dimensions, of which the theorems are those of spherical geometry.

What the non-euclidian and two-dimensional dwellers in the plane would call the same triangle in different positions would be such triangles as had for projections the same spherical triangle in different positions upon the sphere.

In extending these projective measures to geometry of three dimensions, we seek to establish, with reference to a single assumed *central point* in space, a consistent system of projective measurement for all lines and angles however situated in space. In doing this we shall establish for every plane in space its central point, and with reference to this point a system of projective measurement in the plane similar to but not identical with that described above. For this purpose Dr. McClintock distinguishes a *prime* plane in which the projective measurement is defined by means of a tangent sphere whose radius is r, and a *secondary* plane in

* Read before the New York Mathematical Society, March 4, 1893.

which the projective measures are made on a sphere whose radius is still r, but whose centre is at a perpendicular distance p from the central point of the plane. Thus in the prime plane the projective distance is equal to the real distance just at the central point, but for the secondary plane the measures are all less in the ratio r/p (less, for in all cases $p > r$) than they would be if the sphere retaining the same centre touched the plane.

Having obtained these systems of measurement, Dr. McClintock now dismisses the sphere, and, taking all planes which pass through the "centre of space" as prime planes, shows that every other plane is a secondary plane, its central point being of course the foot of the perpendicular upon the plane from the centre of space. In establishing this it is to be noticed that the projective measure of every sect upon every straight line has been already fixed by the fact that the straight line is in a prime plane, that is, a plane passing through the centre. What we wish to show is that all the projective measurements thus established for lines which lie in a new plane not passing through the centre are consistent with one another in forming a system peculiar to the plane, namely, that above described as belonging to a secondary plane, in which, as stated in the BULLETIN, the constant factor (denoted above by r/p) is a simple function of the perpendicular from the centre of space upon the plane. So that "all planes in space are either prime or secondary, and all projective distances and angles in space are parts of one harmonious system of general geometry of three dimensions."

In studying this elegant result, and obtaining the constant factor referred to above, it occurred to me that additional clearness, from a purely geometrical point of view, would result from the following simple modification of the definitions of projective distances:

Assuming a "central point" A and a linear magnitude r, the "projective measure" of a given finite line or "sect" is its apparent magnitude as seen from a point at a distance r from A measured in a direction perpendicular to the plane through the given line and A. There is thus a common *view-point*, as we may call it, or rather a common pair of view-points for all lines lying in a given plane passing through A. We may, of course, instead of the apparent magnitude, which is an angle, take the subtending arc in the circle whose radius is r, but we shall still call this the apparent magnitude. The radius will be always the same, and we are concerned only with the position of the view-point.

Consider now an indefinite straight line. Having found a view-point P as above, it is a view-point for any segment or sect of the line. Pass a plane through A perpendicular to

the line, and therefore also through the two view-points. Call the point O, where this plane cuts the given line, its *centre*. Then it is obvious that any point of the circle drawn through the view-points from this centre will serve equally well as a view-point. That is, from every point of this circle any sect of the line has the same apparent magnitude, which apparent magnitude is its projective measure. We shall call this circle the *view-locus* of the given line. The view-locus is then a circle of which the given line is the axis; its plane passes through A; and, denoting the distance of the line from A by p, its radius is $\sqrt{(p^2 + r^2)}$. Now pass any plane through the given line, and let $AA' = d$ be the perpendicular upon it from A. Call A' the *centre* of this new plane. The extremities P', P' of the chord AA' produced form a pair of view-points for the given line OX, which, with reference to the plane $OA'X$, might have been constructed thus: From the centre A' of the plane erect a perpendicular $A'P' = r'$, where $r'^2 = r^2 + d^2$, [since, if $OA' = p'$, $r'^2 + p'^2 = r^2 + p^2 = r^2 + p'^2 + d^2$]. Now, since the position of A' and the value of r' depend only upon the plane, and not upon the given line OX, we have thus found a pair of view-points which will answer for any line in the new plane as well as for OX. This new plane is Dr. McClintock's secondary plane; its view-points are further from the plane than those of the prime planes, and are not symmetrically situated with respect to the centre of space.

Fig. 1.

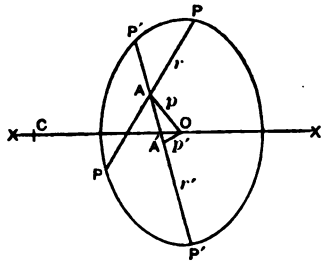
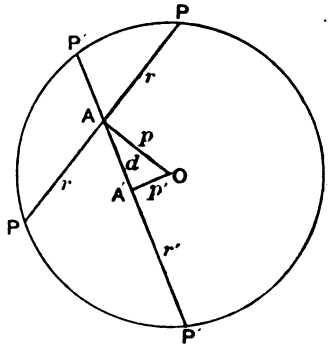


Fig. 2.



This establishes the harmonious and consistent system required. There is for each line a view-locus, and for each plane a definite view-point. The view-point of a plane is on the view-locus of a line in the plane, so that when a line is regarded as belonging to different planes, the system of apparent or projective measurements assigned to it in the different planes is consistent.

The projective measure of an angle may now be defined, once for all, as its apparent magnitude as seen from the view-

point of its plane. The definition is unique, and no similar theorem for consistency exists.

Having constructed the view-points of a given plane, by laying off from A' (the foot of the perpendicular $AA' = d$) $A'P' = r'$, where $r'^2 = r^2 + d^2$, upon the perpendicular; let C be any point of the plane, distant $AC = a$ from the centre of space: then $A'C = a'$, where $a'^2 = a^2 - d^2$. Therefore $CP'^2 = a'^2 + r'^2 = a^2 + r^2$. Since this depends only upon the position of C, it follows that the view-point of every plane passing through C is at a constant distance from C, that is, it lies upon a sphere which we may call the *view-sphere* of C.

The view-locus of a line is a circle having the line as axis, and with reference to which the *power* of A is $-r^2$. The view-sphere of a point is a sphere of which it is the centre, and with reference to which the power of A is $-r^2$, while the view-locus of any diameter is the small circle of this sphere whose plane is perpendicular to it and passes through A.

When two lines intersect, their view-loci are small circles of the view-sphere of the point of intersection, and these circles intersect in the two view-points of the plane of the intersecting lines.

Again, the view-spheres of two points intersect in a circle, viz., the view-locus of the line joining the points. Those of three points not in a straight line intersect in the two view-points of the plane of the three points, and other similar relations might be stated.

THE ADJUSTMENT AND COMPARISON OF OBSERVATIONS.

Lehrbuch der Ausgleichsrechnung nach der Methode der kleinsten Quadrate. Von Dr. K. J. BOBEK. Stuttgart, J. Maier, 1891. 8vo, pp. viii. + 176.

The Theory of Errors and Method of Least Squares. By WILLIAM WOOLSEY JOHNSON. New York, John Wiley & Sons, 1892. 12mo, pp. x. + 174.

THE text-book of Dr. Bobek is a curiosity as regards its catechetical form and typographical arrangement. In its substance, however, it is clear, sound, and highly practical. All the doubtful points regarding deductions of the law of facility of error, probable errors, and criteria for rejection of observations are passed over in silence, and the author in his 32 answers to questions, 29 explanations, and 52 problems writes with a certainty that should tend to inspire the student with confidence. Many lengthy examples of adjustments of observations in geodesy and physics are carried out in full detail with tabular forms for arranging the computations.

Professor Johnson's work is a careful and scholarly text-book on both the theory and practice of the subject. The doubtful points regarding the deduction of the law of error are not considered, and no place is given to criteria for rejection. More than one half the book, however, is devoted to discussions regarding the probability of errors, and to probable errors and comparisons of precision. The investigations regarding the probability surface and the probable errors of target shooting may be noted as one of these discussions, and another is that regarding the deductions of formulas for probable error. It is clearly pointed out that the probable error should be computed from the original individual observations and not from any weighted groups of these. It is shown that the assumption of the arithmetical mean involves the same law of errors for indirect as for direct observations. The way being thus made clear the whole discussion of indirect observations, both independent and conditioned, is given in the 29 pages of chapter VIII., while the last chapter treats of the solution of normal equations.

The tendency of modern text-books on the method of least squares in the direction of avoiding doubts regarding the deduction of the law of facility of error is clearly shown by the two works before us. The arithmetical mean being boldly assumed, and perhaps the assumption justified by a quotation from Gauss or by the discussion of Encke, the

well-known demonstration follows, and the law of facility directly gives the conclusion that the most probable values of the observed quantities are those which render the sum of the squares of the residual errors a minimum, the precision of all the measurements being the same throughout. Some books, indeed, assume this final conclusion at the outset—a method probably more advantageous for certain classes of students than the procedure of Gauss's first proof. Gauss himself in later writings rejected this proof—partly, perhaps, on account of the assumption of the arithmetical mean as the most probable value, and partly, as Bertrand has suggested, on account of the insufficiency of the assumption that $y = \phi(x)$ represents the law of facility; for, the probability of an error depends both on its magnitude and on that of the measured quantity a, so that strictly the law is $y = \phi(a, x)$.

The idea of probable error is often slowly grasped by beginners. Professor Johnson's definition is an excellent one—"The error which is just as likely to be exceeded as not is called the probable error"—and the conception of regarding it as a measure of the risk of error will be of value to students. Nevertheless the tendency of text-books to devote a good deal of space to discussions and computations of probable error, mean error, and mean of all errors is in general to be deplored, as it is apt to convey an erroneous impression concerning the practical comparison of the precision of observations. An observer who thoroughly understands both his instrument and method of measurement is able to give a far better statement regarding the precision of the results than can any computer who deduces probable errors. The observer can with some confidence assign weights for the combination of measurements made at different times, but even he should hesitate to determine weights solely from computed probable errors. The method of least squares, in spite of the uncertainties of its fundamental proofs, is strong and sound when used for the adjustment of observations; notwithstanding the certainties of its reasoning it is often practically weak when used for the comparison of the precision of observations.

MANSFIELD MERRIMAN.

LEHIGH UNIVERSITY, *March* 3, 1893.

ON THE DEFINITION OF LOGARITHMS.

BY PROF. MELLEN W. HASKELL.

AFTER reading Professor Stringham's interesting paper* on "A Classification of Logarithmic Systems," and at his suggestion, I undertook to examine his definitions from the point of view of the theory of functions. The results of my investigation are embodied in the following paper. If I have taken the liberty of deriving some well-known formulæ, I hope that at least the point of view will prove interesting.

Starting from Riemann's definition of the logarithm as that function $\phi(z)$ which satisfies the equation

$$\phi(uz) = \phi(u) + \phi(z), \tag{1}$$

we can easily find an expression for the derivative of the logarithm in terms of the derivative of the independent variable as follows:† Differentiating equation (1) on the assumption that z is constant, we have

$$z\phi'(uz) = \phi'(u);$$

whence, if we write $u = 1$, it follows that

$$z\phi'(z) = \phi'(1).$$

The value of $\phi'(1)$ can be chosen at will and is characteristic for the system of logarithms under consideration. It is called the *modulus* of the system, and we shall denote it by $M = m \operatorname{cis} \beta$.

Let us write $\phi(z) = w$, and denote time-derivates by dots. The last equation obtained can then be written

$$\dot{w} = M\frac{\dot{z}}{z}. \tag{2}$$

Writing $z = r \operatorname{cis} \theta$, we find by differentiation

$$\dot{z} = \dot{r} \operatorname{cis} \theta + r(-\sin\theta + i\cos\theta)\dot{\theta} = (\dot{r} + ir\dot{\theta}) \operatorname{cis} \theta$$

$$= \left(\frac{\dot{r}}{r} + i\dot{\theta}\right)z, \tag{3}$$

whence follows immediately

$$\dot{w} = M\frac{\dot{z}}{z} = M\left(\frac{\dot{r}}{r} + i\dot{\theta}\right). \tag{4}$$

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* *American Journal of Mathematics*, vol. 14, p. 187.

† See, for example, Durège, "Theorie der Funktionen," 3. Aufl.

Representing values of z in the usual way by points in a plane with polar coördinates r and θ, and the corresponding values of w by points in a second plane, if a point P in the z-plane move along any given curve, the corresponding point Q in the w-plane will likewise move along a curve, and the relative velocities of the two moving points can be computed from equations (3) and (4). The character of the two curves will evidently depend entirely upon the quantities $\dot{r}/r$ and $\dot{\theta}$, to which can be assigned any values, constant or variable, that we choose. The simplest case is obviously that in which $\dot{r}/r$ and $\dot{\theta}$ are regarded as constant, and we shall find that this case is exactly the one of most importance for us.

Let $\dot{r}/r = \lambda$ and $\dot{\theta} = \omega$, to bring our notation into conformity with that of Professor Stringham. Then formulæ (3) and (4) become respectively

$$\dot{z} = (\lambda + i\omega)z, \tag{5}$$

$$\dot{w} = M(\lambda + i\omega), \tag{6}$$

and the movement of the points in the two planes may be described as follows: The velocity of P (representing z) is made up of two components, one along the straight line OP joining the origin to P, the other perpendicular to OP, each of them being in magnitude proportional to the length OP. In other words, P moves along OP with velocity proportional to its distance from the origin O, while at the same time OP revolves about O with constant angular velocity ω; that is to say, P traverses a logarithmic spiral with velocity proportional to the radius vector. The velocity of Q (representing w), on the other hand, is constant in amount and direction, and Q will therefore describe a straight line.

It remains for us to determine more closely the relation between the spiral and the corresponding straight line. The quantity $(\lambda + i\omega)$ is absolutely arbitrary, but occurs as a factor in both $\dot{z}$ and $\dot{w}$ and therefore does not affect the *relative* velocity of P and Q. Let us put

$$\lambda + i\omega = \mu \operatorname{cis} \psi,$$

then

$$\dot{w} = m\mu \operatorname{cis} (\beta + \psi) = m\mu \operatorname{cis} \phi.$$

The point Q then moves always upon a line making an angle ϕ with the axis of real quantities, while, as for the logarithmic spiral, the constant angle which it makes with any radius vector is the angle ψ,—the arbitrary angle $\phi - \beta = \tan^{-1} \lambda/\omega$ of Professor Stringham's paper. The position of the spiral will be completely determined as soon as we fix any one point

upon it,—for example, the point at which it crosses the unit circle.

Let us first see what straight line corresponds to the spiral through the point where the axis of real quantities cuts the unit circle. For this point $z = 1$. If in the equation of definition (1) we put $z = 1$, we find

$$\phi(u) = \phi(u) + \phi(1).$$

One value of the logarithm of 1 is therefore 0, and the corresponding straight line in the w-plane is the line *through the origin* which makes an angle ϕ with the axis of real quantities.

In Professor Stringham's paper he obtains a spiral through any given point of the plane by assigning suitable values to the ratio $\lambda : \omega$, letting all the spirals pass through the same point ($z = 1$) of the unit circle. I prefer, however, to retain the same value of this ratio $\lambda : \omega$ and allow the point of intersection with the unit circle to vary. One spiral of this kind will pass through any given point of the plane, and spirals chosen in this way have the obvious advantage that they intersect only at the points 0 and ∞.

We can determine the value of w corresponding to any point of the unit circle by treating the unit circle as a special case of the logarithmic spiral, viz., when $\lambda = 0$. The equations for the velocity of P and Q then become

$$\dot{z} = \omega i z, \qquad \dot{w} = m\omega \operatorname{cis}\left(\frac{\pi}{2} + \beta\right).$$

P then describes the unit circle with uniform angular velocity ω, while Q moves with constant velocity $m\omega$ along the straight line through the origin which makes an angle $\left(\frac{\pi}{2} + \beta\right)$ with the axis of real quantities. The relation between z and w is then evidently this: when $z = \operatorname{cis}\theta$,

$$w = m\theta \operatorname{cis}\left(\frac{\pi}{2} + \beta\right) = \mu i\theta.$$

Now, as P continues to revolve and z passes over and over again through the same values, Q moves continually forward and we obtain an infinite number of values of w corresponding to each value of z. The complete correspondence is given by writing

$$z = \operatorname{cis} \theta,$$

$$w = m(\theta + 2n\pi) \operatorname{cis} \left(\frac{\pi}{2} + \beta\right)$$

$$= Mi(\theta + 2n\pi),$$

where n is any integer, positive, negative, or zero.

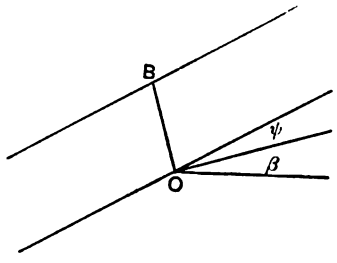
Let us now return to the family of spirals with the same angle ψ, and determine the corresponding lines of the w-plane. To the spiral through the point for which $z = \operatorname{cis} \theta$ corresponds the line through the corresponding w. But as this w has an infinite number of values, so to each spiral correspond an infinite number of *parallel* straight lines through the points for which

$$w = Mi(\theta + 2n\pi).$$

In other words, to every z correspond an infinite number of values of w, which differ from each other only by multiples of $2\pi Mi$. This quantity is then the *modulus of periodicity* of the logarithmic function. It reduces to the familiar form $2\pi i$, when $M = 1$.

A word with regard to the conformable representation. It is evident that we shall obtain a complete representation of the z-plane upon a stripe of the w-plane bounded by any two consecutive lines corresponding to any one of the spirals. Furthermore, as the slope of these lines depends on the angle of the spirals and that is arbitrary, the slope may be anything we like, but the distance between corresponding points of the two boundaries is fixed, since it depends on the modulus alone.

Fig. 3.



Finally, it is worth noticing that if we put $\omega = 0$, the spirals all become straight lines through the origin, the angle $\psi = 0$, the slope of the straight lines in the w-plane reduces to β, and Napier's definition of the logarithm only requires to be modified by fixing the slope of the lines and the distance of one of them from the origin.

BERKELEY, CAL., *December* 10, 1892.

NOTE ON THE DEFINITIONS OF LOGARITHM AND EXPONENTIAL.

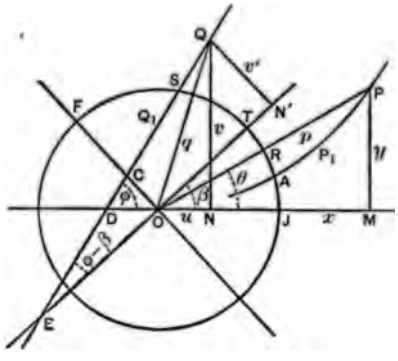
BY PROF. IRVING STRINGHAM.

PROFESSOR HASKELL'S interesting and important investigation calls for a modification of my former definitions of the logarithmic and exponential functions.* I propose that the new specifications shall be as follows:

In a circle whose radius is unity OT is fixed and makes an angle β with the real axis, OR turns about O with a constant speed, Q moves along any line ES in the plane with a constant speed, P along OR with a speed proportional to its distance from O.

Following the notation of my former paper, let the velocities of P, Q, R be:

Fig. 4.



P in OR at $A = \lambda$,
Q in ES $= \mu$,
R in JRS $= \omega$;
and let arc JR $= \theta$,
circular measure of ODS $= \phi$,
$OP, OQ = p, q$,
$ON, NQ = u, v$,
$ON', N'Q = u', v'$,
$OM, MP = x, y$.

In all logarithmic systems the relation

$$\frac{\omega}{\lambda} = \tan(\phi - \beta)$$

is assumed to exist. This fixes the ratio of the radial to the transversal velocity of P and determines the *form* of the curve upon which P moves. Its *position* may then be determined by fixing two points upon it, and for this purpose we may assume that P crosses the real axis OJ at the instant Q crosses the line ET, and that P crosses the circumference of the unit circle at the instant Q crosses the line OF which is drawn through the origin perpendicular to ET. As a consequence of these two assumptions, when ES passes through the origin, A coincides with J, and the figure becomes identical with that of my former paper (*loc. cit.* p. 187), which

* *American Journal Mathematics*, vol. 14, p. 187.

thus appears as a special case of the one here presented. Here also the property that $OP = 1$, when $OQ = 0$, is retained.

Referred to this more general figure my former definitions of modulus, base, logarithm, and exponential (*loc. cit.* p. 188) may now stand without any modification whatever. In somewhat changed phraseology they are:

The modulus is

$$\kappa \equiv \frac{\mu}{\sqrt{\lambda^2 + \omega^2}} (\cos \beta + i \sin \beta)$$

$$= \frac{\mu}{\lambda} \cos (\phi - \beta) (\cos \beta + i \sin \beta)$$

and fixes the system as soon as β has been assigned.

The base is the value that OP, or $x + iy$, assumes at the instant when OQ becomes 1, that is, when Q passes through the point J as it moves along some line that intersects the unit circle at J. Let this base be represented by B.

The ratio OP/OP_1 is the exponential, with respect to base B, of the difference $OQ - OQ_1$:

$$(x + iy)/(x_1 + iy_1) = B^{(u+iv)-(u_1+iv_1)},$$

or with respect to modulus κ:

$$(x + iy)/(x_1 + iy_1) = \exp_\kappa [(u + iv) - (u_1 + iv_1)].$$

In particular, since $x + iy = 1$ when $u + iv = 0$,

$$x + iy = \exp_\kappa (u + iv).$$

Inversely, the difference $OQ - OQ_1$ is the logarithm, with respect to the same modulus or base, of the ratio OP/OP_1:

$$(u + iv) - (u_1 + iv_1) = \log_\kappa (x + iy)/(x_1 + iy_1),$$

or

$$(u + iv) - (u_1 + iv_1) = {}^B\log(x + iy)/(x_1 + iy_1),^*$$

and in particular

$$u + iv = \log_\kappa(x + iy)$$
$$= {}^B\log(x + iy).$$

* It is convenient to have the means of expressing a logarithm both with respect to its base and its modulus. There are good reasons for preferring the German notation, as here indicated, in the former case, reserving the subscript notation to indicate the modulus; and there is now good usage in English for the German practice, in Cathcart's translation of Harnack's *Differential u. Integral Rechnung.*

From these definitions the laws of operation for logarithms and exponentials and the generalized Eulerian formula,

$$B^{u+iv} = |B|^{u\cos\beta + v\sin\beta}\, e^{\frac{i}{m}(v\cos\beta - u\sin\beta)}, \quad (\textit{loc. cit.}, \text{ p. } 192)$$

are easily deduced, only slight modifications of my former proofs being necessary.

The position of the point A is determined as follows: Since by definition $\omega = \lambda \tan(\phi - \beta)$

$$\text{and} \qquad m = \mu/\lambda \,.\, \cos(\phi - \beta),$$

$$\therefore \quad m\omega = \mu \sin(\phi - \beta).$$

But ω and $\mu \sin(\phi - \beta)$ are the rates of change of θ and v' respectively, and $\theta = 0$, $v' = 0$ are simultaneous values;

$$\therefore \quad m\theta = v';$$

and since $\theta = JA$, $v' = OC$ are also simultaneous values;

$$\therefore \quad m.JA = OC.$$

From these data the morphosis (*Abbildung*) in the plane as Professor Haskell has constructed it, is readily obtained.

In view of the fact, first pointed out by Professor Haskell, that, whatever be the base or modulus, we may make P move upon a straight line by making Q move parallel to OT, my proposed distinction between linear and non-linear systems (*loc. cit.*, p. 192) is a false classification. Any system has both characteristics. The true classification is into gonic systems with complex modulus and agonic or ordinary systems with real modulus. In order to determine a gonic system completely, we have only to assign a fixed value to the modulus—its versor part by fixing β, its tensor part by fixing the ratio $\mu/\sqrt{\lambda^2 + \omega^2}$. The elements of this ratio may then vary at pleasure, subject only to the condition here imposed, and to the permanent condition $\omega = \lambda \tan(\phi - \beta)$.

BERKELEY, *January* 10, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, March 4, at half-past three o'clock, the president, Dr. McClintock, in the chair. Mr. Everett Irving Yowel, of the University of Cincinnati, having been duly nominated, and being recommended by the council, was elected a member. Professor W. Woolsey Johnson read a paper entitled "A case of non-euclidean geometry." This paper appears in the present number of the BULLETIN, p. 158.

Professor Johnson also read a note "On the construction of the roots of a quadratic whose coefficients are complex." In this note it was pointed out that, constructing complex quantities in the usual way, if an ellipse be drawn of which the foci are $\pm ca$, and z is a semi-diameter, then $w = \sqrt{(a^2 - z^2)}$ is the conjugate semi-diameter. Conversely if w and z are conjugate semi-diameters of an ellipse, the points $a = \pm \sqrt{(w^2 + z^2)}$ are the foci. Hence, to construct the roots of the quadratic in the form $z^2 - 2pz = q^2$: Lay off p from the origin, and from its extremity $\pm q$. Construct the ellipse of which these are conjugate semi-diameters. The foci of this ellipse represent the roots.

Or, putting the quadratic in the form $z^2 - 2pz + q^2 = 0$: Lay off p from the origin and from its extremity $\pm q$. Construct the ellipse of which these last are semi-diameters and the origin a focus. The extremities of the conjugate diameter represent the roots.

The special case in which the coefficients are real was considered in each construction.

WE have to announce the death of Professor Enrico Betti at Pisa on August 12 last, that of Professor H. Streintz at Graz in the middle of November, and that of F. von Mocnik also at Graz on December 1.

PROFESSOR CLEVELAND ABBE of the U. S. Weather Bureau, Washington, wishes to come into direct communication with American mathematicians who have a personal interest in the problems of aero-dynamics as illustrated in the phenomena of the atmosphere. He would be glad to send to any member of the NEW YORK MATHEMATICAL SOCIETY who may desire it a copy of the *Mechanics of the Atmosphere*, recently published by the Smithsonian Institution. Professor Abbe has also for distribution some copies of Döllen's valuable ephemens for observations in the vertical of Polaris during 1893.

T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APOLLONII PERGÆI quæ græce extant cum commentariis antiquis. Edidit et latine interpretatus est I. L. Heiberg. Vol. II. Leipzig. Teubner, 1893. 8vo. 86 and 362 pp. M. 4.50

CORNELY (A.). Untersuchungen über involutorische Gleichungssysteme. Würzburg, 1892. 4to. 28 pp. M. 1.50

DIOPHANTI ALEXANDRINI opera omnia cum græcis commentariis. Edidit et latine interpretatus est P. Tannery. Vol. I: Diophanti quæ extant omnia continens. Leipzig, Teubner, 1893. 8vo. 9 and 481 pp. M. 5.00

DOEHLEMANN (K.). Über die festen und involutorischen Gebilde, welche einige ebene Cremona-Transformationen enthalten können. München 1892. 8vo. 54 pp. 1 plate. M. 2.00

DYSON (F. W.). The potential of an anchor ring. London 1893. 4to. 58 pp. Ill. M. 3.20

EDWARDS (J.). Differential calculus for beginners. New York, Macmillan, 1893. 16mo. 10 and 262 pp. $1.10

ERLER (W.). Einleitung in die analytische Geometrie und in die Lehre von den Kegelschnitten. 2te Aufl. Berlin, Dümmler, 1893. 4 and 82 pp. 1 plate. M. 1.00

FORNARI (N.). Collezione di poliedri geometrici a superficie sviluppabile. Milano 1893. 32 figure ritagl. su cartonc. forte. M. 3.60

HAENTSCHEL (E.). Studien über die Reduction der Potentialgleichung auf gewöhnliche Differentialgleichungen. Ein Anhang zu Heines Handbuch der Kugelfunctionen. Berlin 1893. 8vo. M. 8.00

HIPPEL (A.). Beitrag zur Bewegung eines Punktes auf einer Kugel. Marburg 1892. 8vo. 40 pp. M. 1.20

HIRSCH (A.). Zur Theorie der irrationalen Differentialgleichungen mit rationalem Integral. Königsberg 1892. 4to. 43 pp. M. 1.50

ILLIGENS (E.). Die unendliche Anzahl und die Mathematik. Münster 1893. 8vo. 60 pp. M. 1.00

JETTMAR (H. v.). Versuch der Einführung homogener Punkt- und Liniencoordinaten in die Elemente der analytischen Geometrie. Wien 1892. 8vo. 45 pp. 3 plates. M. 2.00

LANGHEINEKEN (P.). Das Potential einer materiellen Kugel, deren Dichtigkeit eine ganze rationale Function der rechtwinkligen Coordinaten ist. Leipzig 1892. 8vo. 59 pp. M. 1.50

LÖKLE (F.). Untersuchungen aus der synthetischen Geometrie. Stuttgart 1892. 4to. 79 pp. M. 2.00

MACFARLANE (A.). The imaginary of algebra. Being a continuation of the paper "Principles of the algebra of physics." Salem Press, Salem, 1892. 8vo. 23 pp. [Reprint from the Proceedings of the American Association for the Advancement of Science, vol. 41 (1892), pp. 33–55.]

———. The fundamental theorems of analysis generalized for space. Norwood Press (Cushing), Boston [1893]. 8vo. 32 pp. (Copies can be had from the author, Austin, Texas.) $0.50

MÜLLER (F.). Carl Heinrich Schellbach. Gedächtnissrede gehalten in der Aula des Kgl. Friedrich Wilhems-Gymnasiums am 29sten October 1892. Berlin, Reimer, 1893. 8vo. 35 pp. With Schellbach's portrait. M. 0.50

STAHL (H.) und KOMMERELL (V.). Die Grundformeln der algemeinen Flächentheorie. Leipzig, Teubner, 1893. 8vo. 6 and 114 pp. 1 plate. M. 4.00

STURM (R.). Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung. 2ter Theil: Die Strahlencongruenzen erster und zweiter Ordnung. Leipzig, Teubner, 1893. 8vo. 14 and 367 pp. M. 12.00

ZEIDLER (J.). Untersuchung der in orthogonalen Punktcoordinaten durch die Gleichung $z = \sin x \sin y$ dargestellten Fläche. Leitmeritz 1892. 8vo. 27 pp. M. 1.20

II. ELEMENTARY MATHEMATICS.

CASEY and DOWLING. Elements of Euclid. 6th ed. 1892.

CLARKE (A. D.). Plane trigonometry for the use of students preparing for examinations. London, Longmans, 1893. New and revised edition. 8vo. 362 pp. 5s.

DUPUIS (N. F.). Elementary synthetic geometry of the point, line, and circle in the plane. New York, Macmillan. $1.10

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1ter Theil: Ebene Geometrie. 2te verbesserte Auflage. M. 2.00

2ter Theil: Raumgeometrie. M. 1.50

FLOR. Lösung des Problems: die Quadratur des Kreises. Riga 1892.

GAJDECZKA. Lehrbuch der Arithmetik und Algebra. Leipzig, Freytag, 1892.

GAUSS (F. G.). Fünfstellige volständige logarithmische und trigonometrische Tafeln. 38ste Aufl. Halle 1893. 8vo. 166 and 35 pp. Cloth. M. 2.50

GIAVARINI (C.). Sulla elementare trisezione in parti eguali dell, angolo rettilineo: studio. Modena, Rossi, 1892. 8vo. 14 pp. 1 plate. Fr. 0.25

JACKOWITZ. Hauptsätze der Stereometrie. Schrimm, Schreiber, 1892.

LOCK (J. B.) and SCOTT (C.A.). Arithmetic for schools. New York, Macmillan, 1893. $0.70

OCCELLA (F.). Alcune questioni di matematica elementare. Casale, tip. Casalese, 1892. 8vo. 34 pp.

RATTO (G.). La trisezione dell'angolo, risolta e dimostrata geometricamente. Genova, Montalto, 1892. 4to. 15 pp. 1 plate.

SCHNELLINGER (J.). Fünfstellige Logarithmen für die Zehner-Logarithmen der natürlichen und trigonometrischen Zahlen. Wien 1892. 8vo. 6 and 146 pp. Cloth. M. 3.00

ZETZSCHE (K. E.). Katechismus der ebenen und räumlichen Geometrie. 3te vermehrte und verbesserte Aufl. Leipzig 1893. 8vo. Cloth. Ill. M. 3.00

III. APPLIED MATHEMATICS.

ABBE (C.). The mechanics of the earth's atmosphere: a collection of translations. Washington, D. C., Smithsonian Institution, 1891 [1893]. 3 and 324 pp. [Smithsonian miscellaneous collections No. 843.]

ALBRECHT (M. F.) und VIEROW (C. S.). Lehrbuch der Navigation und ihrer mathematischen Hülfswissenschaften. Für die kön. preussischen Navigationsschulen bearbeitet. 7te Aufl. Berlin 1893. 8vo. 24 and 745 pp. 2 star charts. M. 12.00

ARCE (J. DE). Resistencia de materiales y estabilidad de las construcciones. Vol. I. Madrid 1892. 4to. 16 and 368 pp. M. 11.00

BARKER (G. F.). Physics. Advanced course. 2d ed., revised. New York, Holt, 1892. 8vo. 10 and 902 pp. Cloth. $3.50

BORTKEWITSCH (L. v.). Die mittlere Lebensdauer. Die Methoden ihrer Bestimmung und ihr Verhältniss zur Sterblichkeitsmessung. Jena 1893. 8vo. 6 and 117 pp. M. 3.00

CZAPSKI (S.). Theorie der optischen Instrumente. Breslau 1893. 8vo. 240 pp. 1 plate. Ill. Cloth. M. 8.00

DRION (C.) et FERNET (E.). Traité de physique élémentaire. 12me éd. Paris 1893. 8vo. 4 and 852 pp. Ill. M. 6.80

FAUSTMANN (V.). Didaktische Bemerkungen zur elementaren Mechanik. Czernowitz 1892. 8vo. 39 pp. M. 1.20

FIGUIER (L.). L'année scientifique et industrielle, ou exposé annuel des travaux scientifiques, des inventions, etc. Année 36 (1892). Paris 1893. 12mo. M. 3.00

GALILEI (G.). Le opere; ristampate fedelmente sopra la edizione nazionale sotto gli auspici di S. M. il Re d'Italia. (In 20 vols.) Vol. III, parte 1: Il Sidereus nuncius e le scritture ad esso attinenti. Firenze, Barbèra, 1892. 4to. 399 pp. Ill.

GANOT. Elementary treatise on physics. Translated by E. Atkinson. 14th ed., revised and enlarged. London 1893. 8vo. 1126 pp. Ill. Cloth. M. 15.50

GLAZEBROOK (R. F.). The laws and properties of matter. London 1893. 8vo. Cloth. M. 2.80

LONEY (S. L.). Mechanics and hydrostatics for beginners. 12mo. 272 pp. Pitt Press mathematical series, Cambridge Warehouse. 4s. 6d.

LORENTZ (H. A.). La théorie électromagnétique de Maxwell et son application aux corps mouvants. Leide 1892. 8vo. 194 pp. M. 3.00

TICHY (A.). Die Präcisions-Tachymetrie und ihre neuesten instrumentalen Mittel. München 1893. 8vo. 76 pp. 1 plate. M. 1.50

WITTWER (W. C.). Grundzüge der Molecular-Physik und der mathematischen Chemie. 2te vermehrte und verbesserte Aufl. Stuttgart, Wittwer, 1893. 8vo. 10 and 304 pp. M. 6.00

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Saturday, Jan. 7,	3:30	Saturday, April 1,	3:30	Saturday, Oct. 1,	3:30
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" Mar. 4,	"	" June 6,	"	" Dec. 2,	

The Annual Meeting will be held on Thursday, December 31, at 4:30 P.M.

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INTRODUCTORY MODERN GEOMETRY.

Introductory Modern Geometry of Point, Ray, and Circle. By WILLIAM BENJAMIN SMITH, A.M., Ph.D. (Goett.), Professor of Mathematics and Astronomy, University of the State of Missouri. New York and London, Macmillan & Co., 1893. 16mo, pp. vii + 297.

IN reviewing a book, one of the canons of fair criticism is to regard its adaptation to the readers for whom the author himself designs it; but as a preliminary to this notice, we must object to the selection implied in the preface, where Professor Smith describes his book as intended "to present in simple and intelligible form a body of geometric doctrine acquaintance with which may fairly be demanded of candidates for the Freshman class," and then points out that *one year's study of geometry* is about as much as can be expected in schools. Our own conviction is that geometry may with great advantage be taught to children in their early school days. The simplest kind of geometry, of course; with few formal proofs, and depending more on the teacher than the text-book. But even when this early introduction has been omitted, the subject seems to be one that may be presented in formal guise to the average child of twelve or thirteen. It relates to that with which he is already practically familiar, illustrations may be drawn from his every-day experience, his unconscious perceptions of space relations may be appealed to and formulated; and thus it presents itself as a valuable discipline by which his reasoning faculties may be developed, and his vague disconnected perceptions organized, without burdening him with a mass of new and possibly uninteresting facts.

But just because geometry is so eminently fitted for the youthful mind it should be *at first* presented in such form as to be in accord with the general views of laymen, when these are not in direct opposition to the truth. It may well be that the ideas of modern geometry on such questions as that of the nature of space ought to be explained earlier and to more students than is now done; but if we accept W. K. Clifford's rule, "before teaching any doctrine, wait until the nature of the evidence for it can be understood," such discussions will not be put in the forefront of our geometrical teaching. They will be more easily and profitably treated when they can be founded on knowledge derived from a study of Euclidean space. "There is no time of reading a book better than when you need it, and when you are on the point of finding it out yourself if you were able," says J. Clerk Maxwell; why then should we thrust upon the student "the

consoling hope that, after all, this other [view of space] may be the true state of things," until he is able to appreciate the relief thus offered, through having, with Clifford, suffered from "the dreary infinities of homaloidal space"?

But given this preliminary study of geometry, then comes the time for a concise systematic treatment of the subject with direct reference to modern ideas. Certainly all teachers of even the most elementary geometry ought to know wherein these modern views differ from the older ones; and to those that have neither opportunity nor inclination for abstruse mathematical studies, a book on the lines of the one before us will afford valuable assistance. To some such circle of readers, therefore, we prefer to regard this introductory modern geometry as addressed.

Professor Smith begins with a discussion of the nature of the properties of space, emphasizing Riemann's distinction between *infinite* and *boundless*. The order of development of any system of geometry depends on what we accept as fundamental ideas. Here, following Bolyai, the idea of the *distance* between two points is regarded as fundamental, and the conceptions of plane and straight line are deduced from this. We have also the idea of an *angle;* though the fact that this is just as much a fundamental assumption as the idea of distance is not sufficiently emphasized. The distances, as also the angles, are not regarded as capable of comparison, except as to being equal or unequal. Deduced from these fundamental conceptions we have then some of the ordinary propositions depending on a comparison of the simpler rectilinear figures, leading up to a careful and logical introduction of parallels, pp. 49–56. The exact nature of the assumptions made in the ordinary treatment of parallels is very clearly brought out. We have the theorem, "Either *no* triangle has the sum of its angles equal to two right angles, or *every* triangle has the sum of its angles equal to two right angles." The choice to be made is shown to depend on the answer to the question, Can two indefinite lines be drawn in a plane so as not to meet? And here experience yields no answer; we may adopt either hypothesis, deducing a set of axioms, and on these our system of geometry must be founded. Here the choice having been made, the result is formulated in two axioms expressing that through a given point to a given line one parallel and only one can be drawn.

From this point the geometry is specifically and exclusively the geometry of Euclidean space. Much use is made of the principle of symmetry; and in connection with the two kinds of symmetry, axial and central, the principle of reciprocity is explained. This we should have liked to see introduced much earlier; how simply and naturally it can be done may

be seen in Professor Henrici's admirable little book "Congruent Figures." Properties of symmetrical rectilinear figures having been established, the discussion of the symmetrical curve, the circle, logically follows; and here the vicious practice of so many American text-books now in use, of treating angles in a circle by means of their numerical measure, is departed from, and the original purely geometric method of proof is reverted to. In fact, all numerical properties are left to be treated together in a later section; and following the circle, we have propositions on the relation of areas, and on proportions in general, all with the satisfactory rigorous geometric treatment. A useful idea, rarely found in elementary works, is introduced—that of the generation of an area by the motion of a line; the use of this is possible, since the fact that an area, just as much as a line or an angle, is susceptible of sign is here explained. After this we have the taction problem of Apollonius, in whose date there is a misprint, 200 A.D. being given instead of 200 B.C. In the section on metric geometry the transference of numerical propositions to geometry is clearly and scientifically explained; and six or seven pages are devoted to the trigonometrical ratios, just sufficient to excite the student's interest in trigonometry.

This outline will show how logical is the sequence of the contents; the arrangement of the arguments and demonstrations will be found equally pleasing. And yet the book, with all its good points, is hopelessly marred by the author's persistent disregard of conventional nomenclature. If he had entirely invented the science, or at any rate this special development of it, we could not deny him the legal right of naming his own creations, even though we might deplore his unfortunate choice of such terms as *tract* (finite straight line), *perigon* (four right angles), *numeric*, used as a noun, and *finity*, in contradistinction to infinity. But all the ideas here set forth are the common property of the mathematical world; the science of geometry was not invented yesterday, and it is already provided with a fairly complete and satisfactory English vocabulary. It requires very cogent reasons to justify a scientific writer in deviating from established usage in nomenclature or notation. Why speak of the base circle in reciprocation as the *referee?* Why speak of the point of *tangence* rather than of the point of contact? Why, in short, disfigure the work with the multitude of strange and uncouth terms that we find here? Have not the most effectual reformers been conservative in trifles?

To some of Professor Smith's specialties we may object on even stronger grounds; as, for example, to his notation for sin α and cos α; to his employment of the word *dimension* for the actual length of the base (or altitude) of 'an areal

figure'; and to his absolute use of the term *double point*, speaking of a double point in a correspondence as if it were a double point on the circle considered, and then defining consecutive points as points that "fall together in a double point."

If Professor Smith could be induced to translate his work into ordinary mathematical English, we feel sure that he would greatly increase its usefulness not only in aiding "the first upward steps of the climber," but also in preparing him for the "much higher ascent."

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE, PA., *March*, 1893.

WRONSKI'S EXPANSION.

BY PROF. W. H. ECHOLS.

IN 1810 Höené Wronski presented to the French Academy of Sciences the following formula, without demonstration,

$$fx = a_0 + a_1\omega_1 + a_2\omega_2 + \ldots \text{ ad. inf.}, \quad . \quad . \quad . \quad (1)$$

in which fx, ω_1, ω_2, . . . are arbitrary functions of x, and a_0, a_1, . . . are independent of x. This formula, or rather the law for the formation of the coefficients, he called *la loi suprême.*

Lagrange and Lacroix were appointed as a committee to examine Wronski's memoir and to report on it to the Academy. This report is an admirable production and in every way worthy of the distinguished names attached to it. It is especially noticeable for its conservative tone and yet its acknowledged recognition of the importance and possible future of the formula. The commissioners must have been very much impressed, to have repeated section III. in section IV., "Ce qui a frappé vos commissaires dans le mémoire de M. Wronski, c'est qu'il tire de sa formule toutes celles que l'on connait pour le développement des fonctions, et qu'elles n'en sont que des cas très-particuliers." It would seem remarkable in view of this that nothing has been done toward developing his work and placing it on a sound scientific basis. The whole of Wronski's work and method of work appears to be purely qualitative; it is truly algorithmic, inasmuch as he

uses all symbols of quantity with as little regard for their quantitative relations as though they were purely operational symbols without quantitative properties. When one reads Wronski's published writings, one should read them with this understanding, not for the positive results, but for the suggestions which the analytical forms hold out. When we remember that the work of Cauchy and Abel had yet to be done in clearing up this matter of series, it was well that the clear head of Lagrange was called into service in this matter.

The report, while in every way complimentary to Wronski, was misconstrued by him and he did not avail himself of the excellent suggestions made by the commissioners. He took offence, which showed itself in his "Mémoire sur la réfutation de la théorie des fonctions analytiques de Lagrange."

In the following year he presented a second memoir to the Academy, submitting his *problème-universel des mathématiques*, a particular case of *la loi suprême*, upon which was based subsequently his so called refutation of the theory of functions of Lagrange. This was also presented without demonstration. It was referred to a committee for examination which was composed of Arago and Legendre. This report does not deal so kindly with Wronski, and throughout his life he never recovered from the bitterness which it engendered in him against the authors and even the Academy itself. These two reports are given in full in the "Supplément à la Réforme de la Philosophie," p. lxxxvi. I prefer to consider the more generous and liberal opinion of Lagrange and Lacroix, rather than the severe one of Arago and Legendre, as indicating the existence of truth and promise of future utility in the unfortunate Wronski's undemonstrated formula.

In 1811 Wronski published his "Introduction à la Philosophie des Mathématiques," quarto, pp. 270. This is chiefly occupied with the metaphysical foundation of mathematics and the classification of its orders and branches. In 1812, after the report on his first memoir, he published the "Réfutation de la Théorie des Fonctions, etc.," and in 1815 was published the "Philosophie de la Technie Algorithmique," quarto, pp. 286. This last is the most important of his writings. The work opens with the words, "En quoi consistent les mathématiques ?—N'y aurait-il pas moyen d'embrasser, par un seul problème, tous les problèmes de ces sciences, et de résoudre généralement ce problème universel ? Telles sont les questions qui doivent couronner la philosophie du géomètre; et telles sont aussi les questions qu'à la fin de notre Philosophie des Mathématiques, sous la marque (XXXII.), nous avons embrassées dans notre loi algorithmique absolue, comme nous allons le montrer." The formula marked (XXXII.) is (1) at the head of this paper.

Wronski unhesitatingly lays down such general laws of form as

$$Fx = A_1\phi_1 x + A_2\phi_2 x + \ldots,$$
$$Fx = (A_1 + \phi_1 x)(A_2 + \phi_2 x)\ldots,$$
$$Fx = A_0 + \cfrac{\phi_1 x}{A_1 + \cfrac{\phi_2 x}{A_2 + \ldots}}$$

for arbitrary functions ϕ.

The last ninety pages of the Philosophie de la Technie are devoted to the so-called demonstration of la loi suprême, which consists in determining a form for the coefficients. The major portion of the nincty pages are taken up with the development of his "shinn" functions (Wronskians).

The problème-universel is expressed in the formula

$$0 = fx + x_1 f_1 x + x_2 f_2 x + \ldots,$$

and upon it Wronski founded his "Résolution générale des equations algébriques de tous les degrés." In 1817 the Academy of Sciences of Lisbon proposed as the subject of a prize the demonstration of Wronski's formula. The prize was in the following year awarded to M. Torriani for the refutation of it.

In 1847 were published the "Messianisme" and "Réforme du Savoir Humain," which contain the "Réforme des Mathématiques," "Résolution des equations algébriques," "Réforme de la Philosophie," etc.

The literature on Wronski and his methods consists of: a paper by Professor Cayley in the *Quarterly Journal* (1873) "On Wronski's Theorem," which brought forth two others in the *Nouvelles Annales*, for April and July 1874, by M. Abel Transon. In the *Comptes Rendus* (April 1881), by M. Yvon Villarceau under the section Mécanique Céleste,—"Note sur les Methodes de Wronski. "Exposé des Méthodes Générales en Mathématiques, d'après Höené Wronski," by M. Emile West (Gauthier-Villars, 1886). A small pamphlet published in 1890 by the same firm, "Loi Téléologique (12 pp.) précédée d'un Autobiographie et Inventaire de l'Œuvre (62 pp.)." The autobiography is taken from the "Réforme de la Philosophie." In *Bibliotheca Mathematica*, 1892, No. 2, a short note of five pages, "Sur les découvertes Mathématiques de Wronski," by S. Dickstein, gives a list of the titles of Wronski's works and of the papers by other writers which refer to them. The author of this note has in preparation a complete bibliography of the published works and manuscripts of Wronski, which will be published by the Academy of Sciences

of Cracow and will form the basis for the edition of Wronski's mathematical works which the Academy proposes to publish. Wronski's manuscripts are in the library of Kornik (near Posen), and are the property of Count Zamoyski. In the preface to his "Exposé," M. West says: "Wronski's works are very rare, and it is difficult to read them at first. In this work we propose to make known new mathematical processes which are applicable to the most difficult questions in mathematics. The methods in question, while put forth in the beginning of the century by a mathematician whose name was disgraced under circumstances useless to recall, remain even to-day unknown. Höené Wronski has been dead for thirty years, and the causes which turned attention away from his work no longer exist; our generation is not interested in the old quarrels over the truly remarkable works of a learned and profound mathematician."

At the close of his article on Wronski's methods in celestial mechanics, M. Villarceau says: "Those who wish to undertake a similar work we advise to consult, if possible, the manuscripts left by Wronski. Judging them by the works he has published, one will be sure to find a precious mine of analytical developments, very correctly executed."

Höené Wronski, born in Posen, August 24, 1778, died in Neuilly, August 9, 1853. At sixteen years of age he was an officer of artillery under Kosciusko. He was made prisoner by the Russians at the battle of Maciecowicz, October 10, 1794, and at the close of the Polish revolution he entered the Russian army and reached therein the grade of Lieutenant-Colonel. He resigned in 1797 to devote himself exclusively to the study of philosophy and the mathematical sciences, studied two years in Germany, and then in France, where he remained until his death.

Those who have written upon Wronski and his methods are enthusiastic upon the subject, but none of them seems to have gone down into his work or to be able to do more than write with high praise of its character and seek to draw the attention of others to his writings. Now that the Academy of Cracow has taken the matter in hand we may hope that the question as to the real merit of Wronski's work will be settled. When this has been done, if the writer may offer his impression acquired through a somewhat cursory examination, it is quite likely that it will be found that Wronski's work is mainly formal, or, as I have said, qualitative. None the less is the honor due him for it, for it is the law of mathematical evolution that qualitative analysis shall precede the quantitative, and it is rare that the same mind possesses equally the power of imagination in mathematical theory and realization of its application. Wronski in the highest degree

possessed that mind which, "having seen a leaf and a drop of water, can construct the forests, the rivers, and the seas."

LA LOI SUPRÊME.

It has been frequently remarked that Wronski gives a demonstration of the *loi suprême* in the "Philosophie de la Technie." This is wrong, for Wronski never shows the possibility of the equality sign in (1). What he has done amounts to this: he has assumed the infinite series to be true; upon this assumption he proceeds to determine a form for the constant coefficients. This he does in the last ninety pages of the "Philosophie de la Technie." It is in the course of this analysis that he develops the "combinatory sums" (determinant functions) now known (after Muir) by his name. In the supplement to M. West's "Exposé" one finds the deduction of the coefficient forms after Wronski's own method, which is much abbreviated by the use of modern determinant forms. The most modern form of this determination can be found in Laurent's "Traité d'Analyse," vol. III., p. 352, where the general coefficient is given as follows:

$$a_n = \frac{\Omega_n(fa)}{\Omega_n(\omega_n)} - \sum_{\mu=n+1}^{\mu=\infty} a_\mu \frac{\Omega_n(\omega_\mu)}{\Omega_n(\omega_n)}, \tag{2}$$

wherein some arbitrary value a has replaced the variable x, and

$$\Omega_n(F) = \begin{vmatrix} \omega'_1 & \cdots & \omega'_{n-1}, & F' \\ \cdot & \cdot & \cdot & \cdot \\ \omega^n_1 & \cdots & \omega^n_{n-1}, & F^n \end{vmatrix};$$

the symbol ω^r meaning that after differentiating ω r times x is changed to a. It is to be observed, however, that Wronski in his work does not confine himself to the operation of differentiation, but works out his results by use of an arbitrary symbol of repetitive operation, of which Δ and D are to be regarded as merely particular cases.

The value of any coefficient in the series (1) is thus given by (2) in terms of an infinite series involving all the coefficients which follow it. We may, however, by successive substitution, express a_n in an infinite series, distant terms of which involve coefficients of as high an order as we choose. From the nature of this expression the value of a_n is thus indeterminate inasmuch as we can have no means of testing the arithmetical equivalence of such a series. It would seem

that form (2), or the modified form of it obtained through substitution, leads to no solution of the problem in view.

If we make the upper limit of μ finite, say $m > n$, it is easy to show for small values of m, and doubtless it can be readily shown for large values of m, that (2) may be converted into the proper form for the coefficient, which is

$$a_n = (-1)^n \frac{\begin{vmatrix} \omega' & \dots & \omega'_{n-1} & \omega'_{n+1} & \dots & \omega'_{m-1} & f'a \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega_1^{m-1} & \dots & \omega_{n-1}^{m-1} & \omega_{n+1}^{m-1} & \dots & \omega_{m-1}^{m-1} & f^{m-1}a \end{vmatrix}}{\begin{vmatrix} \omega'_1 & \dots & \omega'_{m-1} \\ \cdot & \cdot & \cdot \\ \omega_1^{m-1} & \dots & \omega_{m-1}^{m-1} \end{vmatrix}}. \quad (3)$$

The only reason that can be assigned for Wronski's not giving this general form to his coefficient is that he could not see at that time how to evaluate this ratio when m became infinite, this being dependent (when the determinants become difference-products) on Jacobi's theorem for writing out the minor of a simple alternant in terms of the sums of the products of the elements taken with repetition, etc. (see *Annals of Mathematics*, vol. 7, Nos. 4, 5); yet the only particular case of (2) which leads to correct results is evident from (3), and does not require Jacobi's theorem, so that it is strange that with all his ingenuity and fertility of resource he did not make use of it.

If the ω functions in (1) be such that ω_q^r vanishes for $r < q$, either through the operation performed on the functions or through the substitution of a specific value for the variable, then the last column in $O_n(\omega_\mu)$, $(\mu = n + 1 \dots \infty)$ vanishes, as does also the second term in the right of (2), leaving for the value of a_n,

$$a_n = \begin{vmatrix} \omega'_1, & 0 \dots 0, & f' \\ \omega''_1, & \omega''_2 \dots 0, & f'' \\ \cdot & \cdot & \cdot \\ \omega''_1, & \omega''_2 \dots \omega''_{n-1}, & f^n \end{vmatrix} \div \begin{vmatrix} \omega'_1, & 0 \dots 0 \\ \omega''_1, & \omega''_2 \dots 0 \\ \cdot & \cdot \\ \omega''_1, & \omega''_2 \dots \omega''_n \end{vmatrix}, \quad (4)$$

which is the form of the coefficient in (1) giving the theorem known as Wronski's expansion. If here we put $\omega_n = (\phi x)^n$ and let $\phi(a) = 0$, we deduce Burmann's series, which including Lagrange's, Laplace's, and therefore Taylor's series was doubtless the cause of section III. in the report of Lagrange and Lacroix upon Wronski's memoir.

If in (1) the ω functions be such that ω_q^r vanishes when $r > q$, then we have from (2)

$$a_n = \frac{f_a^n}{\omega_n^n} - \sum_{\mu=n+1}^{\mu=\infty} a_\mu \frac{\omega_\mu^n}{\omega_n^n},$$

$$= \frac{f_a^n}{\omega_n^n} - \sum_{\mu=1}^{\mu=\infty} \frac{f^{n+\mu} a}{\prod_{r=1}^{\mu} \omega_{n+r}^{n+r}}. \tag{5}$$

This is wrong, for the correct value of a_n in this case must be $f^n a/\omega_n^n$. In particular, if $\omega^q = x_q$, then $a_0 = f0$, and

$$a_n = \frac{f^n x}{n!} - \frac{f^{n+1} x}{n!(n+1)!} - \cdots,$$

which should be the coefficient $f^n x/n!$ in Bernouilli's series.

The value (4) under the conditions imposed is correct, and gives that particular case mentioned in the BULLETIN, vol. 2, p. 140. It seems that throughout Wronski's whole work he has aimed at the generalization given in the BULLETIN as quoted above, which I have called a *composite*.

If inferences may be drawn while the investigations are yet incomplete, this composite may prove useful; and if so, it is my sincere hope that it may be the means of lifting in some measure the weight of opprobrium from the memory of the unfortunate Wronski, whose pathetic story appeals so strongly to the sympathy of all of his co-workers.

CHARLOTTESVILLE, *February*, 1893.

NOTE ON THE SUBSTITUTION GROUPS OF SIX, SEVEN, AND EIGHT LETTERS.

BY F. N. COLE, PH.D.

A LIST of the groups of six, seven, and eight letters is given by Mr. Askwith in vol. 24 of the *Quarterly Journal of Mathematics*, and Professor Cayley has revised and tabulated Mr. Askwith's results in vol. 25 of the same journal.* Noticing

* A list of groups as far as ten letters was given by Kirkman in the *Proceedings of the Literary and Philosophical Society of Manchester*, vol. 3 (1864), p. 144.

that several familiar groups were missing in this table, I have re-examined the whole question by an independent method, with the result that I am able to furnish here a supplementary list of over forty omitted groups of these degrees. The precautions which I have taken to insure accuracy give me a considerable degree of confidence that my results are correct and complete.

I. *Six and Seven Letters.*

1. For six letters the intransitive and multiply transitive groups are correctly given in Professor Cayley's enumeration. The three following non-primitive groups are, however, omitted:

Order 36. $36_3 =$

+	+	+	+	−	−
1,	*abc*,	*abc . def*,	*ab . de*,	*ad . be . cf*,	*adbecf*,
	acb,	*abc . dfe*,	*ab . df*,	*ad . bf . ce*,	*adbfce*,
	def,	*acb . def*,	*ab . ef*,	*ae . bd . cf*,	*adcebf*,
	dfe,	*acb . dfe*,	*ac . de*,	*ae . bf . cd*,	*adcfbe*,
			ac . df,	*af . bd . ce*,	*aebdcf*,
			ac . ef,	*af . be . cd*,	*aebfcd*,
			bc . de,		*aecdbf*,
			bc . df,		*aecfbd*,
			bc . ef,		*afbdce*,
					afbecd,
					afcdbe,
					afcebd.

Order 24. $24_5 =$

+	−	+	−	+	−
1,	*ac*,	*ac . bd*,	*ac . bd . ef*,	*abe . cdf*,	*aebcfd*,
	bd,	*ac . ef*,		*abf . cde*,	*abecfd*,
	ef,	*bd . ef*,		*ade . cbf*,	*abfcde*,
				adf . cbe,	*adecbf*,
				aeb . cfd,	*adfcbe*,
				aed . cfb,	*aedcbf*,
				afb . ced,	*afbced*,
				afd . ceb,	*afdcbe*.

Order 12. $12_2 =$

+	+	+	+
1,	*ac . bd*,	*abe . cdf*,	*aeb . cfd*,
	ac . ef,	*abf . cde*,	*afb . ced*,
	bd . ef,	*ade . cbf*,	*aed . cfb*,
		adf . bec,	*afd . bce*.

The first of these is a subgroup of Cayley's $(abcdef)_{72}$; its systems of non-primitivity are a, b, c and d, e, f. It consists of all the even substitutions which permute the letters of

each system separately and of the *odd* substitutions which interchange the two systems.

The second and third groups are subgroups of Cayley's $(abcdef)_{48}$. Their systems of non-primitivity are a, c; b, d; and e, f. The former consists of all the substitutions which permute the letters of the three systems separately, together with the cyclical permutations of the systems. The latter consists of the even substitution of the former, and is a common subgroup of all the three transitive groups of order 24 with six letters.

2. In the case of seven letters one intransitive group and one doubly transitive group are omitted by Professor Cayley:

Order 24. $24_a =$

1,	1,	$ab\,.\,cd$,	$ac\,.\,bd$,	$ad\,.\,bc$,
efg,	abc,	bdc,	adb,	acd,
egf,	acb,	bcd,	abd,	adc,
ef,	ab,	cd,	$adbc$,	$acbd$,
gh,	bc,	ad,	$acdb$,	$abdc$,
eg,	ac,	bd,	$adcb$,	$abcd$,

where each substitution of e, f, g is to be multiplied by the four substitutions of a, b, c, d which stand opposite it.

Order 168. A doubly transitive group obtained by combining Cayley's $+ (abcdef)_{24} =$

1,	$ab\,.\,cd$,	$abe\,.\,cdf$,	$abcd\,.\,ef$,
	$ac\,.\,bd$,	$abf\,.\,cde$,	$adcb\,.\,ef$,
	$ad\,.\,bc$,	$ade\,.\,bfc$,	$aecf\,.\,bd$,
	$ae\,.\,cf$,	$adf\,.\,bec$,	$afce\,.\,bd$,
	$af\,.\,ce$,	$aeb\,.\,cfd$,	$bedf\,.\,ac$,
	$bd\,.\,ef$,	$aed\,.\,bcf$,	$bfde\,.\,ac$,
	$bf\,.\,de$,	$afb\,.\,ced$,	
	$be\,.\,df$,	$afd\,.\,bce$,	

with

$$abcefdg.$$

This is the well-known triad group of seven letters, discussed by Kronecker and others, which occurs in connection with the transformation of the seventh order of the elliptic modular functions. It is certainly remarkable that this group among all others should have been overlooked.

II. *Eight Letters.*

1. The following 20 intransitive groups are not given by Prof. Cayley:

Order 96. $\{(abcd)$ all $(efgh)$ all$\}$sex. =

1,	$ab\,.\,cd$,	$ac\,.\,bd$,	$ad\,.\,bc$,	1,	$ef\,.\,gh$,	$eg\,.\,fh$,	$eh\,.\,fg$,
abc,	bdc,	adb,	acd,	efg,	fhg,	ehf,	egh,
acb,	bcd,	abd,	adc,	egf,	fgh,	efh,	ehg,
ab,	cd,	$adbc$,	$acbd$,	ef,	gh,	$egfh$,	$ehfg$,
ac,	bd,	$adcb$,	$abcd$,	eg,	fh,	$ehgf$,	$efgh$,
ad,	bc,	$acbd$,	$abdc$,	eh,	fg,	$eghf$,	$efhg$.

Order 72. $\{(abcdef)_{72}\ (gh)\}$dim.$_3$,
$(abcdef)_{36_2}\ (gh)$.

There are three dimidiates of $(abcdef)_{72}$ and (gh), since the former has three subgroups of order 36.

Order 48. $\{(abcdef)_{48}\ (gh)\}$dim.$_3$,
$(abcdef)_{24_5}\ (gh)$.

Order 36. $\{(abcdef)_{36_3}\ (gh)\}$dim.$_1$,
$(abcdef)_{36_2}\ (gh)$ dim.$_2$,
$\{(abc)$ all (def) cyc.$\}(gh)$.

Order 24. $\{+(abcdef)_{24}\ (gh)\}$dim.,
$\{\pm(abcdef)_{24}\ (gh)\}$dim.,
$\{(abcdef)_{24_5}\ (gh)\}$dim.,
$(abcdef)_{12_2}\ (gh)$.

Order 16. Six groups based on a two-to-two isomorphism between $(abcd)_8$ and $(efgh)_8$. They are

1,	$ac\,.\,bd$,	1,	$eg\,.\,fh$,
ac,	bd,	ef,	fh,
$ab\,.\,cd$,	$ad\,.\,bc$,	$ef\,.\,gh$,	$eh\,.\,fg$,
$abcd$,	$adcb$,	$efgh$,	$ehgf$,

where the last three lines on either side may be permuted in any of the six possible ways.

Order 12. $A\ \{(abcd)$ pos. $(efgh)$ pos.$\}$ =

1,	$ab\,.\,ef$,	$abc\,.\,efg$,	$abcd\,.\,efgh$,
	$ac\,.\,eg$,	$acb\,.\,egf$,	$adcb\,.\,ehgf$,
	$ad\,.\,eh$,	$abd\,.\,efh$,	$abdc\,.\,efhg$,
	$bc\,.\,fg$,	$adb\,.\,ehf$,	$acdb\,.\,eghf$,
	$bd\,.\,fh$,	$acd\,.\,egh$,	$acbd\,.\,egfh$,
	$cd\,.\,gh$,	$adc\,.\,ehg$,	$adbc\,.\,ehfg$.
		$bcd\,.\,fgh$,	
		$bdc\,.\,fhg$,	

Order [illegible]. A group based on a one-to-two isomorphism between $(abcd)$ and $(efgh)$:

1,	1,	$eg \,.\, fh$,
$ab \,.\, cd$,	eg,	fh,
$ac \,.\, bd$,	$ef \,.\, gh$,	$eh \,.\, fg$,
$ad \,.\, bc$,	$efgh$,	$ehgf$.

2. Of the non-primitive groups there are also 20 missing:

Order [illegible]. $(abcd)$ all $(efgh)$ all pos. $(aebf \,.\, cg \,.\, dh)$.

This is composed of all the even substitutions which affect the letters of the systems a, b, c, d and e, f, g, h separately, together with the odd substitutions which interchange the two systems.

Order [illegible]. I find four groups of this order, while Cayley gives only two. The four are

$$(abcd) \text{ all } (efgh) \text{ all sex. } (ae \,.\, bf \,.\, cg \,.\, dh),$$
$$(ab)(cd)(ef)(gh) \text{ all } (ABCD)_{12},$$
$$-[(ab)(cd)(ef)(gh) \text{ pos. } (ABCD) \text{ all}],$$
$$\pm[(ab)(cd)(ef)(gh) \text{ pos. } (ABCD) \text{ all}].$$

Here A, B, C, D represent the systems $a, b; c, d; e, f; g, h$. In the third group the substitutions of A, B, C, D are even in a, b, c, d, e, f, g, h, while in the fourth group the substitutions of A, B, C, D are even or odd in a, b, c, d, e, f, g, h, according as they are even or odd in A, B, C, D.

Order 128. $(abcd)$, $(efgh)$, $(ae \,.\, bf \,.\, cg \,.\, dh)$.

Order 96. $(ab)(cd)(ef)(gh)$ pos. $(ABCD)$ pos.

Order 64. $(abcd)$, cyc. $(efgh)$, cyc. dim. $(aecg \,.\, bf \,.\, dh)$,
$(abcd)$, pos. $(efgh)$, pos. dim. $(aebfcgdh)$,
$(abcd)$, com. $(efgh)$ com. dim. $(ae \,.\, bf \,.\, cg \,.\, dh)$,
$(abcd)$ com. $(efgh)$, com. dim. $(aebfcgdh)$.

Order 48. $(ab \,.\, cd \,.\, ef \,.\, gh)(ABCD)$ all.

This is obtained by combining

$$ABC = ade \,.\, bcf, \quad ABCD = acfhbdeg.$$

Order 32. Four groups obtained by combining

1,	$ac \,.\, bd$,	1,	$eg \,.\, fh$,
ac,	bd,	eg,	fh,
$ab \,.\, cd$	$ad \,.\, bc$,	$ef \,.\, gh$,	$eh \,.\, fg$,
$abcd$,	$adcb$,	$efgh$,	$ehgf$.

with one of

$$ae \,.\, bf \,.\, cg \,.\, dh, \quad aebf \,.\, cgdh, \quad aecg \,.\, bfdh, \quad aebfcgdh.$$

Two groups

$$\left\{\begin{array}{rr|rr} 1, & ac\,.\,bd, & 1, & eg\,.\,fh, \\ ac, & bd, & ef\,.\,gh, & eh\,.\,fg, \\ ab\,.\,cd, & ad\,.\,bc, & eg, & fh, \\ abcd, & adcb, & efgh, & ehgf, \end{array}\right\}(ae\,.\,bf\,.\,cg\,.\,dh),$$

$$\left\{\begin{array}{rr|rr} 1, & ac\,.\,bd, & 1, & eg\,.\,fh, \\ ac, & bd, & efgh, & ehgf, \\ ab\,.\,cd, & ad\,.\,bc, & ef\,.\,gh & eh\,.\,fg, \\ abcd, & adcb, & eg, & fh, \end{array}\right\}(ae\,.\,bf.\,cg\,.\,dh).$$

Order 24. $A\ \{(abcd)$ pos. $(efgh)$ pos.$\}(ae\,.\,bf\,.\,cg\,.\,dh)$.

$$(ab\,.\,cd\,.\,ef\,.\,gh)\ (ABCD) \text{ pos.}$$

The latter is obtained by combining

$$ABC = ade\,.\,bcf, \quad AC\,.\,BD = afbe\,.\,chdg.$$

Order 16. Two groups obtained by combining

$$\begin{array}{llll} 1, & ac\,.\,ef\,.\,gh, & ab\,.\,cd\,.\,ef\,.\,gh, & abcd\,.\,efgh, \\ & bd\,.\,eh\,.\,fg, & & adcb\,.\,ehgf, \\ & ab\,.\,cd\,.\,fh, & & \\ & ad\,.\,bc\,.\,eg, & & \end{array}$$

with one of the two substitutions

$$aebfcgdh, \quad afbgchde.$$

3. Of the multiply transitive groups Professor Cayley omits two:

Order 56. $\{A\ (abcdefgh), \quad bcedghf\}$.

Order 168. There are two doubly transitive groups of this order:

$$\{A(abcdefgh), \quad (bcedghf)_{21}\},$$
$$\{(abcdefg)_{21}, \quad (ad\,.\,be\,.\,ch\,.\,fg)\}.$$

The latter is given by Mr. Askwith, and is the *simple* group of 8 letters. It is isomorphic with the group of the same order in 7 letters. Professor Cayley's notation (p. 154) is misleading, inasmuch as there is no cyclical substitution of 8 letters in either group, both being composed of even substitutions.

The two groups of order 168 are discussed by Hölder, *Math. Annalen*, vol. 40, p. 75.

One misprint in Professor Cayley's list remains to be noted. The group 4, of 8 letters should read:

$$\{(abcd)_4, \quad (ef)(gh)\} \text{ quad.}$$

The corrected total number of groups of degrees 2–8 is as given below:

No. of letters ..	2,	3,	4,	5,	6,	7,	8.
No. of groups ..	1,	2,	7,	8,	37,	40,	199.
Transitives	1,	2,	5,	5,	16,	7,	48.

ANN ARBOR, *April*, 1893.

MATHEMATICAL BIBLIOGRAPHY.

Revue semestrielle des publications mathématiques rédigée sous les auspices de la Société mathématique d'Amsterdam par P. H. SCHOUTE (Groningue), D. J. KORTEWEG (Amsterdam), W. KAPTEYN (Utrecht), J. C. KLUYVER (Leyde), P. ZEEMAN (Delft), avec la collaboration de MM. C. VAN ALLER, F. DE BOER, J. CARDINAAL, D. COELINGH, R. J. ESCHER, W. MANTEL, P. MOLENBROEK, P. VAN MOURIK, M. C. PARAIRA, A. E. RAHUSEN, G. SCHOUTEN, J. W. TESCH, J. VERSLUYS, J. DE VRIES et de Madlle. A. G. WIJTHOFF. Tome I. (Première partie). Amsterdam, W. Versluys, 1893. [8vo, 104 pp. 4 florins or 8.50 francs per year.]

AT the international congress for mathematical bibliography held at Paris, in July 1889, under the auspices of the French mathematical society, a detailed classification was adopted to serve as a basis for a general bibliography of the mathematical sciences. It was resolved to prepare a *répertoire bibliographique* of the mathematical literature of the present century (1800–1889) arranged by subjects in logical order. This bibliography was to be continued by supplements issued at intervals of ten years.

The execution of this work was entrusted to a permanent commission composed of five French members (H. Poincaré, D. André, Humbert, d'Ocagne, Charles Henry) and twelve foreign members (Catalan, Bierens de Haan, Glaisher, Teixeira, Holst, Valentin, Em. Weyr, Guccia, Eneström, Gram, Liguine, Stephanos). It is not known to the present writer what progress may have been made with this undertaking which will no doubt require several years for its completion.

The *Revue semestrielle*, whose complete title is given above, is a new bibliographical journal of the current periodical literature of mathematics based on this classification. The number now at hand (part 1 of vol. 1) covers the period from March 1 (why not April 1?) to October 1, 1892, and is supposed to have appeared January 1, 1893; the second part will report the period from October 1, 1892, to March 1, 1893, and is to appear

July 1, 1893. If it be found practicable to keep up this prompt appearance of the journal,—and the large number of collaborators would seem to make this possible,—it will prove a feature of decided superiority over the German *Jahrbuch über die Fortschritte der Mathematik*, of which only the first part of the volume for 1890 has just been announced as ready. While it is to be regretted that the non-periodical literature is entirely excluded from the *Revue semestrielle*, it is probably only owing to this limitation of the scope of the work that it becomes possible to issue it so promptly.

Another very wise restriction that the editors have imposed upon themselves consists in giving, not reviews of the papers and memoirs recorded, but, in addition to the name of the author and the title, nothing but a brief synopsis of the contents, generally in not more than 5–10 lines. These synopses are written in French, German, or English, in conformity with the language of the paper reported; for all papers written in languages different from English and German, French is used for the synopsis, while the title is of course always given in the original language.

The arrangement is not by subjects, as in the *Jahrbuch*, but by countries and journals, the countries being given in alphabetical order, so that America leads. At the end we find a list of the periodicals recorded, a subject index, and an alphabetical index of authors. The subject index deserves special mention. It is not alphabetical, but based on the classification of mathematics adopted by the Paris congress of 1889. This classification is given on pp. 91–99, with references by page numbers to the papers recorded in the preceding pages. Every paper is marked with one or more symbols indicating the subject or subjects of which it treats and thus assigning its place in the classification.

This whole system if once understood is very simple and convenient. But a note explaining the meaning of the various letters and numbers used might well have been added for the benefit of the reader not familiar with the system. It must be remembered that the plan of classification adopted by the Paris congress of 1889 has never before been applied in practice on a large scale and is therefore little known to the general mathematical public. Moreover it differs widely from the more familiar classification used by the *Jahrbuch*. This difference is very much to be regretted. The general plan followed by the *Revue* for recording the current mathematical literature appears to the writer to be decidedly preferable to the plan adopted in the *Jahrbuch*. It is perhaps too much to hope for a radical change of system in a publication which has done such excellent service for over twenty years. But it may be allowed to express a *pium desiderium*. Assuming

that the *Revue* is firmly established and will continue its work as promised, it would seem best if the *Jahrbuch* would leave the whole range of *periodical* literature entirely to its younger rival and devote itself in the future, i.e., beginning with the year 1892, exclusively to the far more difficult work of recording and reviewing the non-periodical literature of mathematics. It would then be best to do this in the main according to the plan of the Paris congress of 1889.

A few minor objections to this plan brought forward by Eneström (*Bibliotheca mathematica*, new series, vol. 4 (1890), pp. 41–42) and Favaro (*Rivista di matematica*, vol. 1 (1891), p. 72) seem to have been overcome satisfactorily by the editors of the *Revue semestrielle*.

The list of journals and transactions of societies (120 entries) given on pp. 87–90 is perhaps as complete as might reasonably be expected. The editors of periodical publications not yet represented in the list are requested to communicate with M. P. H. Schoute, Groningen.

We miss in particular the journals of the Russian mathematical societies at Moscow and Odessa and the *Vierteljahrsschrift der Naturforschenden Gesellschaft* at Zürich. Perhaps it might also be advisable to include such more technical journals as the *Civilingenieur* and *The Electrician*, as they frequently contain papers of great value, if not in pure mathematics at least in theoretical mechanics and mathematical electricity, subjects that are professedly included in the scope of the *Revue*.

The members of the Amsterdam Mathematical Society, who have undertaken the arduous task of compiling this record of the mathematical literature of the day, deserve the thanks of every working mathematician. The new journal cannot fail to prove extremely useful to the ever increasing number of those engaged in mathematical research.

It would be captious criticism to find fault with occasional slight inelegancies of expression in the English and German notices, or with the few misprints and inaccuracies to be found in this number. In general the work is remarkably well done; the typographical execution is pleasing, and the proof, considering the difficulty of the constant use of at least three languages, is read carefully.

ALEXANDER ZIWET.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, April 1, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. George Lincoln Manning, Stevens Institute of Technology; Mr. Charles Moen Rice, Worcester, Mass.; Dr. Edward Burr Van Vleck, Middletown, Conn. Professor T. H. Safford read a paper entitled "Instruction in mathematics, as affected by modern algebraic and geometrical developments and by the progress of the physical sciences." Mr. Maclay read a note on the "Geometry of the maxima and minima of a certain class of functions."

WE have received through the courtesy of Professor George Bruce Halsted a circular recently issued by the Physico-Mathematical Society of the University of Kasàn, calling attention to the approaching centennial anniversary of the birth of Lobachevsky. It is proposed either to establish a prize bearing the name of Lobachevsky to be awarded for investigations in mathematics, especially in branches appertaining to his work, or to place his bust in the buildings of the University; or, if sufficient funds are obtained, to do both. Lobachevsky was born on October 22, 1793. He studied at the University of Kasàn, and became a professor there in 1812, being the first of its own students to attain that honor. He was made rector in 1827, and he held that position as well as his professorship until 1846. The work of Lobachevsky marks an important epoch in the history of mathematics. He was the first to perceive that Euclid's postulate is equivalent to the assumption of certain properties of space which can be verified only by experiment or observation, and to show the possibility of a geometry without making use of it. Subscriptions to the Lobachevsky memorial fund should be addressed to the Physico-Mathematical Society, Kasàn, Russia.

AMONG the recent announcements of Longmans, Green, and Co. we notice a "Treatise on Electricity and Magnetism," by G. W. De Tunzelmann, 2 vols. A. and C. Black announce a "History of Astronomy during the Nineteenth Century," by Agnes M. Clerke, third edition, revised and enlarged; and "Algebra, an Elementary Text-book for the Higher Classes of Secondary Schools and Colleges," by Prof. George Chrystal, part I., third edition. The Clarendon Press will soon have ready: "Mathematical Papers of the late Prof. Henry

J. S. Smith, with memoir and portrait, 2 vols.; "A Supplementary Volume to Prof. Clerk Maxwell's Treatise on Electricity and Magnetism," by Prof. J. J. Thomson; "A Treatise on the Kinetic Theory of Gases," by Dr. H. W. Watson, new edition; and "An Elementary Treatise on Pure Geometry," by J. W. Russell.

IN Macmillan and Co.'s list are: "An Elementary Treatise on Modern Pure Geometry," by R. Lachlan; "The Theory of Functions," by Prof. Frank Morley and Prof. James Harkness; "An Elementary Treatise on Theoretical Mechanics," by Prof. Alexander Ziwet, part I., Kinematics (the second part will be ready next fall, and the third and last part a year later); "A Short Course in the Theory of Determinants," by Prof. L. G. Weld; a translation of Weisbach's "Mechanics of Hoisting Machinery," by K. P. Dahlstrom; and "Textbook on Electromagnetism and the Construction of Dynamos," by Prof. D. C. Jackson. The same firm will shortly receive from the Cambridge University Press a work on the "Theory of Functions of a Complex Variable," by Dr. A. R. Forsyth.

COPIES of Professor A. Macfarlane's "Principles of the Algebra of Physics," 1892, 53 pp., can be obtained from the author (Austin, Texas, U. S. A.); price $1.00. This paper is a reprint from the *Proceedings of the American Association for the Advancement of Science*, vol. 40 (1891), pp. 65–117.

DR. EDWARD PHRAGMÉN has been appointed to succeed Madame Sophie Kovalevsky as professor of mathematics at the University of Stockholm. T. S. F.

AT the University of Missouri during the first semester of this academic year the following graduate courses in mathematics were given: theory of equations and quantics, solid analytical geometry, differential equations, elliptic functions, mathematical seminary for the theory of probability, practical astronomy. The first two courses were given by Professor Tindall, the last by Professor Updegraff, and the others by Professor W. B. Smith. During the second semester the courses are being continued without change, except in the case of the seminary, which is devoted to hyperspace.

W. B. S.

B. G. TEUBNER, of Leipzig, announces as in preparation: (1) a complete edition of Hermann Grassmann's mathematical works, to be published in three volumes under the auspices of the Saxon Academy of Sciences; and (2) an edition of Leopold Kronecker's works, under the auspices of the Prussian Academy of Sciences. During the last years of his

life, Kronecker himself had planned and made some preparation for the publication of all his writings in a collected form. Soon after his death the Prussian Academy of Sciences, of which Kronecker had been a member for over thirty years, decided to carry out this plan and entrusted its execution to Dr. Karl Hensel, of the University of Berlin. Two series are contemplated, one to comprise Kronecker's published writings (*Gesammelte Werke*), in four volumes, 4to; the other to contain his hitherto unpublished lectures at the University of Berlin (*Mathematische Vorlesungen*), in five volumes, 8vo. The first volume of the *Werke*, which will be adorned with a portrait of the author, is devoted to the pure theory of numbers and the theory of algebraic equations; the second and third volumes will bring the papers on the arithmetic theory of the algebraic functions; the fourth is to comprise the papers on the integral calculus, the theory of elliptic functions, the theory of the potential, mathematical physics, etc. Volumes II. and III. are nearly ready for the printer; volumes I. and IV. will follow later.

The publication of Kronecker's university lectures cannot fail to arouse great interest. For, apart from their intrinsic value as systematic expositions of certain branches of mathematical science by one of the greatest of modern mathematicians, they will serve in a large degree as a commentary on Kronecker's original papers, which are well known to be frequently so concise as to present serious difficulties to those not very familiar with his profound and original ideas. The first volume of the *Vorlesungen*, on simple and multiple integrals, is edited by Dr. E. Netto, and will appear this summer. The remaining volumes (II.: determinants, III.: theory of numbers, IV. and V.: theory of algebraic equations) are in charge of Dr. Karl Hensel, of Berlin, who requests all former students of Prof. Kronecker to assist him in the preparation of these lectures by transmitting to him (Berlin, N. W., Klopstockstrasse 30, II.) for temporary use any careful notes of Kronecker's lectures that they may possess. A. Z.

CORRECTION. Professor O. Bolza sends us the following in connection with his recent article in the BULLETIN: "Professor Camille Jordan has kindly called my attention to a misstatement in my review of Cole's Translation of Netto's Theory of Substitutions, in the February number of the BULLETIN, viz., the assertion at the end of the article (p. 106) that the determination of all primitive solvable groups of order p^k (p a prime) was, in the case $k > 1$, as yet an unsolved problem. The problem has, in fact, been completely solved by Mr. Jordan in book IV. of his 'Traité des Substitutions.'"

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BÖKLE (C.). Die Ordnungscurven des Nullsystems, ihre Vertheilung im Raume und ihre Construction. Frankenthal, 1892. 8vo. 26 pp. Ill. Mk. 1.20

BORGMEYER (J.). Geometrische Untersuchung über den Ort der Fusspunkte der Lote, welche von einem Punkte auf die Strahlen einer linearen Congruenz gefällt werden. Hildesheim, 1893. 8vo. 51 pp. Mk. 1.50

CAUCHY (A.). Œuvres complètes. 1re série. Vols. VIII and IX: Extraits des Comptes rendus de l'Académie des Sciences. Paris, Gauthier-Villars, 1893. 4to. Each volume Fr. 25.00

——— ——— 2e série. Vol. X.: Résumés analytiques de Turin. Nouveaux exercices de Prague (Mémoire sur la dispersion de la lumière). Paris, Gauthier-Villars, 1893. 4to. Fr. 25.00

GÉRARD (L.). Sur la géométrie non-Euclidienne. Paris, 1892. 4to 115 pp. Ill. Mk. 4.50

GOUTZMER (A.). Ueber gewisse partielle Differentialgleichungen höherer Ordnung. Halle, 1893. 4to. 22 pp. Mk. 1.20

JORDAN (C.). Cours d'analyse de l'École polytechnique. (In 3 vols.) Vol. I. Calcul différentiel. 2e édition, entièrment refondue. Paris, Gauthier-Villars, 1893. 8vo. 18 and 612 pp. Ill. Fr. 17.00

KRAFT (F.). Abriss des geometrischen Kalküls. Nach den Werken Hermann Grassmanns bearbeitet. Leipzig, Teubner, 1893. 8vo. 12 and 255 pp. Ill. Mk. 6.00

LACHLAN (R.). An elementary treatise on modern pure geometry. New York, Macmillan, 1893. 12mo. 10 and 288 pp. $2.25

LUCAS (E.). Récréations mathématiques. Vol. III. Paris, Gauthier-Villars, 1893. 8vo. Fr. 6.50

MARTONE (M.). Introduzione alla teoria delle serie. Parte II. Catanzaro, 1892. Fr. 5.00

MINKOWSKI (H.). Geometrie der Zahlen. Eine Festschrift zum siebzigsten Geburtstage von Herrn Charles Hermite. Leipzig, Teubner, 1893. 8vo.

SCHELLENBERG (C.). Neue Behandlung der hypergeometrischen Function auf Grund ihrer Definition durch das bestimmte Integral. Göttingen, 1892. 8vo. 68 pp. Mk. 1.20

WILLINK (A.). The world of the unseen. An essay on the relation of higher space to things eternal. New York, Macmillan, 1893. 12mo. 184 pp. $1.25

II. ELEMENTARY MATHEMATICS.

BELIKOV (A.) and NATHING (A.). Text-book of algebra, with a collection of exercises. [Russian.] St. Petersburg, 1893. 8vo. 6 and 402 pp. Mk. 6.00

CLARK (A. D.). Army, Woolwich, and Civil Service readers: geometrical problems selected from the examination papers of the last twenty years. New York, Longmans, 1893. 8vo. 63 pp. Cl. $0.60

GRÉVY (A.). Compositions données depuis 1872 aux examens de Saint-Cyr. Algèbre et géométrie. (Énoncés et solutions.) 2d ed. Paris, Gauthier-Villars, 1893. 8vo. Ill. Fr. 2.50

HALL (W. S.). Mensuration. Boston, Ginn, 1893. 6 and 62 pp. 12mo. (Mathematical series.) Cl. $0.55

IPATA (L.). Del circolo e delle sue proprietà trigonometriche. Roma, Perino, 1892. 8vo. 16 pp. 3 plates.

LAISANT (C. A.). Recueil de problèmes de mathématiques, classés par divisions scientifiques. Vol. I. Arithmétique, algèbre élémentaire, trigonométrie. Paris, 1893. 8vo. Fr. 2.50

LEONHARDT (G.). Grundzüge der Trigonometrie und Stereometrie. Halle, 1893. 8vo. Ill. Mk. 1.20

MICHELSEN (P.). Die bestimmten algebraischen Gleichungen des 1sten bis 4ten Grades. Nebst Anhang: Unbestimmte Gleichungen. Hannover, 1892. 8vo. 8 and 306 pp. Mk. 4.00

MILNE (W. J.). Elements of arithmetic for primary and intermediate classes in public and private schools. New York, American Book Company [1893]. 12mo. 240 pp. Cl. $0.30

NEUMANN (K. W.). Lehrbuch der allgemeinen Arithmetik und Algebra. 6th ed. Bremen, Heinsius, 1892. 215 pp. Mk. 2.80

PETERSEN (J.). Analytisk Plangeometri. 3d ed. Part 1. Kjöbenhavn, 1893. 8vo. 96 pp. Mk. 2.50

PINCHERLE (S.). Algebra complementare. Parte I: Analisi algebrica. Milano, Hoepli, 1893. 16mo. 8 and 182 pp. Ill. Mk. 1.50

REMBACZA (M.). Krétko zebrana historya geometryi wykreslnéj. [Brief history of descriptive geometry.] Part II. Stanislau, 1891. 8vo. 64 pp. Mk. 2.00

RUSSELL (J. W.). An elementary treatise on pure geometry, with numerous examples. London, Frowde, 1893. 8vo. 338 pp. 10*s.*6*d.*

WILLS (H.). Handbook of algebra. London, 1893. 8vo. 268 pp. Cl. 3*s.*6*d.*

III. APPLIED MATHEMATICS.

APPELL (P.). Traité de mécanique rationnelle. (In 3 vols.) Vol. I: Statique. Dynamique du point. Paris, Gauthier-Villars, 1893. 1er fasc., 320 pp. Fr. 14.00

BARLOW (C. W. C.) and BRYAN (G. H.). Elementary mathematical astronomy. With examples and examination papers. London, Clive, 1893 [Univ. Corr. Coll. Tutorial Ser.]. 8vo. 446 pp. 8*s.*6*d.*

BAUSCHINGER (J.). Untersuchungen über den periodischen Kometen 1889 V (Brooks). Part I: Definitive Bahnbestimmungen der Hauptkometen aus den Erscheinungen 1889 bis 1891. München, Franz. 4to. Mk. 5.00

BRUGGER (C.). Die Erhaltung der Energie, das Grundprincip der neuen Naturlehre. Einsiedeln, 1893. 4to. 36 pp. Mk. 2.20

DEMARÇAY (J.). Théorie mathématique des guillotines et obturateurs centraux droits. Paris, 1892. 8vo. 4 and 59 pp. Ill. Mk. 1.80

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NEUMANN (C.). Beiträge zu einzelnen Theilen der mathematischen Physik, insbesondere zur Elektrodynamik und Hydrodynamik, Elektrostatik und magnetischen Induction. Leipzig, Teubner, 1893. 8vo. 8 and 314 pp. Ill. Mk. 10.00

RASMUSSEN (A. H. M.). Læren om Skibsdampmaskinen. 2te omarbejdede udgave. Udgiven ved Marineministeriets foranstaltning. Kjöbenhavn, 1893. 8vo. 588 pp. 33 plates. Mk. 11.00

RUDOLPH (H.). Anwendung der akustischen Methode der Längenmessung zur Bestimmung der linearen Ausdehnungscoefficienten von Stäben unter Benutzung einer einfachen Vorrichtung zur Ermittelung der Aenderung der Schwingungszahl einer Saite. Marburg, 1892. 8vo. 49 pp. 1 plate. Mk. 1.20

SEELIG (E.). Molekularkräfte. Physikalisch-chemische Studie der verschiedenen Körperzustände. 2d ed. Berlin, Friedländer, 1893. Mk. 2.40

VÖLKER (H.). Die Centralbewegung. Entwicklung der einschlägigen Gesetze und Zusammenstellung der auf ihnen beruhenden Versuche und Erscheinungen. Berlin, 1893. 4to. 36 pp. 2 plates. Mk. 1.50

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WINKELMANN (A.). Handbuch der Physik. Herausgegeben unter Mitwirkung von F. Auerbach, F. Braun, S. Czapski, K. Exner, u. A. (In 3 vols.) Vol. III.: Elektrizität und Magnetismus. Part 1. Breslau, Trewendt, 1893. 8vo. 546 pp. Ill. Mk. 15.00

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Notes.

New Publications.

Articles for Insertion should be addressed to the

Bulletin of the New York Mathematical Society,

11 East 49th Street, New York City.

The following dates have been fixed for the regular meetings of the Society during 1893:

	P. M.		P. M.		
Saturday, Jan. 7,	8:30	Saturday, April 7,	8:30	Saturday, Dec.	[illegible]
" Feb. 4,	"	" May 6,	"	" Nov.	[illegible]
" Mar. 4,	"	" June 4,	"	" Dec.	[illegible]

BULLETIN

OF THE

NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY

THOMAS S. FISKE. HAROLD JACOBY. ALEXANDER ZIWET.

VOL. II. No. 9.

JUNE, 1893.

NEW YORK:
PUBLISHED FOR THE SOCIETY
BY MACMILLAN & CO.
1893.

THE MECHANICS OF THE ATMOSPHERE.

The Mechanics of the Earth's Atmosphere. A collection of Translations. By CLEVELAND ABBE. City of Washington: published by the Smithsonian Institution, 1891. 8vo, pp. 324.

THE student who in these days sets out to explore any field of mathematical physics, with a view to extending its boundaries, assumes an arduous task. He must, in the first place, become familiar with a certain number of facts, or phenomena, which constitute the basis of his science. He must then learn how these facts are expressed in mathematical symbols, and along what lines the processes of deduction may be applied. This introduction to the mere elements of the science will often find him deficient in some branch of analysis, to which reference must be had before intelligent progress can begin. When this deficiency is supplied, new difficulties are almost certain to arise in the novelty of the conceptions required and in the complexity of the specifications needed for generality. Sooner or later in the course of his studies he learns that the beginnings of the science, as well as the latest conclusions concerning it, are to be found in certain original memoirs. The mere number and volume of these are generally appalling enough. But additional difficulties are commonly met in the fact that they are published in several languages and bound up in the bulky transactions of learned societies. To reach and appreciate them, the student must have access to one or more of the great libraries, and possess at least a reading knowledge of two or three foreign languages. But the obstacles he must encounter and overcome do not end here. He will find that the field of his special science joins on or laps over the boundaries of other sciences, so that he must trace out and define numerous analogies and finally end with a pretty thorough knowledge of several sciences before he is competent to advance any one. The time and energy required to attain this preliminary equipment are very great, and those who do not possess special facilities, along with exceptional talents and unflagging industry, can hardly hope to rise above the plane of mediocrity.

In view of these obstacles in the way of all but the most favored students, we must welcome every effort which seeks the unification and simplification of allied sciences or renders the sources of knowledge concerning them more accessible. Most noteworthy and commendable in the latter regard are the pains recently taken by the French government to republish in collected form the works of Lagrange, Laplace, Fourier, Cauchy, etc., thus enabling the student to place himself in

intimate contact with the methods and appliances of these masters in mathematical and physical research. Private and public enterprise and the zeal of learned societies have also done much of late in the same direction in Germany and England, as shown in the collected works of Bessel, Kirchhoff, Sir George Stokes, Lord Kelvin, Maxwell, and others. The readiness with which such works can now be obtained by the majority of students is one of the most hopeful signs for future progress. Not less noteworthy and commendable are those collections wherein a series of memoirs on the same or allied subjects are brought together in one language. Taylor's Scientific Memoirs of a half century ago are familiar examples of such important aids to readers of English; and the more recent Collection de Mémoirs relatifs à la Physique, publiés par la Société Française de Physique, are similar worthy examples to readers of French.

The collection of translations before us falls in line with the class of works just mentioned, and it is a pleasant duty to thank Professor Abbe and the administration of the Smithsonian Institution for what amounts, almost, to a systematic treatise on the mechanics of the earth's atmosphere, as presented by some of its most recent expounders.

The volume contains twenty papers, all but two of which were published originally in the German language. The subjects embraced in these papers are so inviting to mathematicians generally as well as to those who would bring meteorology within the domain of mathematical physics, that we think it not out of place here to give a full list of the titles and authors. The list is then as follows:

I. HAGEN, 1874. The measurement of the resistances experienced by plane plates when they are moved through the air in a direction normal to their planes.

II. HELMHOLTZ, 1858. On the integrals of the hydrodynamic equations that represent vortex motions.

III. HELMHOLTZ, 1868. On discontinuous motions in liquids.

IV. HELMHOLTZ, 1873. On a theorem relative to movements that are geometrically similar, together with an application to the problem of steering balloons.

V. HELMHOLTZ, 1888. On atmospheric motions. First paper.

VI. HELMHOLTZ, 1889. " " " Second paper. On the theory of wind and waves.

VII. HELMHOLTZ, 1890. The energy of the billows and the wind.

VIII. KIRCHHOFF, 1869. The theory of free liquid jets.

IX. OBERBECK, 1877. On discontinuous motions in liquids.

X. OBERBECK, 1882. The movements of the atmosphere on the earth's surface.

XI. OBERBECK, 1882. On the Guldberg-Mohn theory of horizontal atmospheric currents.

XII. OBERBECK, 1888. On the phenomena of motion in the atmosphere. First paper.

XIII. OBERBECK, 1888. On the phenomena of motion in the atmosphere. Second paper.

XIV. HERTZ, 1884. A graphic method of determining the adiabatic changes in the condition of moist air.
XV. BEZOLD, 1888. On the thermodynamics of the atmosphere. First paper.
XVI. BEZOLD, 1888. On the thermodynamics of the atmosphere. Second paper.
XVII. BEZOLD, 1889. On the thermodynamics of the atmosphere. Third paper.
XVIII. RAYLEIGH, 1890. On the vibrations of an atmosphere.
XIX MARGULES, 1890. On the vibrations of an atmosphere periodically heated.
XX. FERREL, 1890. Laplace's solution of the tidal equations.

The first paper, by Professor G. H. L. Hagen, deals chiefly with experimental data. It was published in 1874, and in addition to its intrinsic merits it is noteworthy for what we would now pronounce a decided demerit in its use of antiquated units.

The second paper, by Helmholtz, is his classic memoir published originally in *Crelle's Journal*. Readers of English have hitherto had access to this memoir through a translation of Professor P. G. Tait published in the *Philosophical Magazine* for June, 1867. But no one can doubt the propriety of republishing it in the present list; for besides marking a distinct advance in hydromechanics, it is at once the most natural and most readable exposition extant of the profound questions considered.

The following five papers, also by Helmholtz, afford a good illustration of the fact alluded to above, namely, that one's foundation for research in any science must be broad as well as deep. They have to deal primarily with the atmosphere, but they present numerous analogies with and allusions to the sciences of thermodynamics, electricity, magnetism, attraction, etc.

The paper by Kirchhoff, No. VIII. of the collection, finds its proper place as an extension to No. III. by Helmholtz. Both Nos. III. and VIII. are of interest in pure mathematics as affording illustrations of the use of conjugate and discontinuous functions for the expression of natural phenomena.

The first paper by Professor Oberbeck gives a very interesting account of his experiments in forcing jets of colored liquid into quiescent colorless liquids. The observed effects, illustrated by numerous meridian sections of the jets, are specially fruitful in suggestions to those who would extend the theoretical investigations of Helmholtz and Kirchhoff. The following papers, X. to XIII., by Oberbeck, deal particularly with the phenomena of the motions of our atmosphere. They are admirable specimens of the application of the principles of hydromechanics to the interpretation of such phenomena, by reason especially of the clear specification of the assumptions

made. We may question the adequacy of these assumptions, but we are not left in doubt as to the limitations they impose on the problems presented by nature. These papers are also of interest and value from the independent confirmation they afford of many results reached previously by Ferrel.

The paper by Dr. Hertz, No. XIV., and those by Bezold, Nos. XV. to XVII., as their titles indicate, deal almost exclusively with the thermodynamics of the atmosphere. While the paper by Hertz is limited to problems requiring the assumption of adiabatic changes only, those of Bezold seek to outline processes which do not rest on so narrow a basis, and which will therefore enable us to understand more clearly the actual phenomena. Both authors are evidently admirers of the graphical method of illustrating mathematical relations, so that those who think best by the aid of diagrams, as well as those who can do without them, will find these investigations attractive. The observer in meteorology, as distinguished from the theorist, will also recognize in Bezold's papers, especially, some fine opportunities for the application of good observational data.

Those who might be deterred from entering the rather difficult field of mathematical meteorology will find encouragement in the words of Lord Rayleigh in the opening paragraph of his paper, No. XVIII., on vibrations of an atmosphere. "In order," he says, "to introduce greater precision into our ideas respecting the behavior of the earth's atmosphere, it seems advisable to solve any problems that may present themselves, even though the search for simplicity may lead us to stray rather far from the actual question. It is proposed here to consider the case of an atmosphere composed of gas which obeys Boyle's law, viz., such that the pressure is always proportional to the density. And in the first instance we shall neglect the curvature and rotation of the earth, supposing the strata of equal density are parallel planes perpendicular to the direction in which gravity acts." He also assumes that the acceleration of gravity is constant through the height considered.

These would seem to be quite simple assumptions, yet they lead to some very interesting results, with which one may well become familiar before passing to the considerations required by the actual atmosphere. The most suggestive of the results referred to are the natural vibration periods of Rayleigh's ideal atmosphere. These are, for the first two harmonics, 23.8 hours and 13.7 hours, respectively, and they may be important in throwing light on the relative amplitudes of the observed diurnal and semi-diurnal changes of atmospheric pressure.

The paper by Rayleigh is fitly followed by that of Margules,

who conceives the periodic variations of pressure in the atmosphere to be due to periodic changes of its temperature. His investigation rests on a foundation of assumptions, which seem nearly, if not quite, adequate. The clearness of presentation and the comprehensiveness of statement render this one of the most important of recent contributions to dynamical meteorology.

Margules' discussion of the case presented by a rotating sphere leads to an application of the differential equation of the second order first treated by Laplace in his theory of tides. The determination of the constants of integration of this equation is a matter of controversy, the principal contestants being Laplace and Lord Kelvin on the one side and Airy and Ferrel on the other. This fact accounts for the unexpected appearance at the end of the collection of the paper by Ferrel on Laplace's solution of the tidal equations. May we not hope that some readers of the BULLETIN will clear up the difficulties and obscurities surrounding this important question?

In a work of this kind one would hardly expect the translator to vary much from a literal rendition. We might wish that he had in the present case taken larger liberties and pruned off some of the verbose indirectness of the originals; but, on the other hand, the literalness serves to keep the reader close to the thought of the author, and this advantage may properly offset the desire for directness in translations so free as these from ambiguity.

The typography of the book is in general good, and we must commend Professor Abbe's effort to secure conformity with the best models of mathematical notation. His use of the "round *d*" to indicate partial differentiation is especially gratifying in view of the perplexing and aimless diversity in this respect of most works in the English language. We have noticed a considerable number of such typographical errors as it is difficult to keep out of a first edition of a book containing a large number of formulas. We may confidently expect that the need of a new edition of the work will afford an opportunity for correcting these inevitable blemishes.

In closing this hasty and quite inadequate sketch of these papers, we would wish to join in the hope expressed by Professor Abbe, "that a coming generation of American meteorologists may prosecute to further conquests the mathematical studies begun by Ferrel and perfected by our European colleagues."

R. S. WOODWARD.

WASHINGTON, *April*, 1893.

THE APPLICATION OF MATHEMATICS TO THE THEORY OF ECONOMICS.

Mathematical Investigations in the Theory of Value and Prices. By IRVING FISHER, Ph.D., Yale College. New Haven, 1892. 8vo, pp. 124. (Reprinted from the *Transactions of the Connecticut Academy*, vol. IX, July, 1892.)

WITH the advance of any branch of human knowledge more refined methods of investigation become necessary. As the fundamental ideas assume more definite shape and the principles relating to them find more accurate expression, not only do more powerful methods of analysis become applicable to the problems which present themselves for study, but they become essential to their solution. It is not surprising, then, that students of economics should be making use more and more largely of mathematics in the development of their subject. In fact, the quantitative character of the principal economic notions, labor, land, capital, price, and utility, seems to point to mathematical methods as particularly appropriate to the subject; and it is found that the relations between some of these elements is of a character quite as dependent upon the operations of pure mathematics as are those between acceleration, velocity, and distance in dynamics. If many have regarded with disfavor the application of mathematical methods to economic science, and have distrusted the information thus obtained, they have been victims of a prevalent confusion between pure theory and its application, and have based their judgment upon a misinterpretation or a misapplication of the analytical results. It must be remembered that a purely theoretical result requires many modifications before it can apply to a practical example, and generally can apply only in an approximate manner.

Up to the present time the principal advantages derived from the use of mathematics have been in the way of illustration. Diagrams and formulas have been employed to give greater definiteness to the ideas under consideration, and although not generally essential to the discussions, have materially assisted in abridging or elucidating them. A literature, however, has already begun to exist and promises to rapidly increase in importance, in which mathematical methods applied to economic phenomena play a *rôle* of the highest interest.

The first work in which the mathematical character of the subject and the utility of mathematical methods received adequate recognition was published by Cournot in 1838, under the title "Recherches sur les principe mathématiques de la

théorie des richesses." * Jevons furnished a most important contribution in 1871 in his "Theory of Political Economy." † This was followed in 1874 by Walras's "Éléments d'économie politique pure," ‡ and in 1889 by the "Untersuchungen über die Theorie des Preises," § of Auspitz and Lieben. In 1890 appeared an interesting and comprehensive treatise by Marshall, "Principles of Economics," ‖ vol. I., with mathematical foot-notes and appendix, a second edition of which followed in the same year. In none of these works has any mathematics of great difficulty been employed. Nothing more complicated than a problem in maxima and minima of a function of several variables, or the discussion of a simple differential equation of the first order, is to be found. Still, much occurs that is of great interest to the student of mathematics.

A recent addition to the literature of this subject is the paper whose title appears at the head of this article. It further clears up some of the ideas already introduced, supplies some new contributions, briefly discusses the work already done in the same line, and gives a valuable bibliography of previous publications.

One of the most interesting features of the progress already made is the introduction of new units appropriate for the measurement of the quantitative conceptions which are peculiar to economics. The introduction of suitable units is just as necessary here as in the case of physical theory, and without such units important advances on mathematical lines would be altogether impossible. The two most important quantitative conceptions yet introduced are utility and marginal utility. The most careful scientific analysis of these conceptions that has come to the writer's notice is contained in the first few pages of Dr. Fisher's paper. We shall follow closely the lines which he has laid down. We do not begin with complete definitions, but we are led by a series of connected definitions to a full appreciation of the nature of the entities considered and to their adequate quantitative definition.

First we use the term utility in a general way, to refer to that property of any commodity by virtue of which an individual is led to desire its possession. The utility of a certain amount of a commodity tends to make an individual willing to forego a certain amount of some other commodity. We lay down the preliminary definition: *For a given in-*

* Paris, Hachette. 8vo, pp. 212.
† London, Macmillan & Co. 8vo, pp. 267.
‡ Lausanne, Rouge. 8vo, pp. 547.
§ Leipzig, Duncker & Humblot. Pp. 550.
‖ London, Macmillan & Co. 8vo, pp. 770.

dividual, at a given time, the utility of a units of A exceeds the utility of b units of B if the individual has a preference for a to the exclusion of b rather than for b to the exclusion of a. This definition is analogous to that which is the basis of inequality in the case of any other quantities, e.g., of forces. If two forces opposite in direction act upon the same particle, that is the greater in the direction of which motion is generated. It follows from this definition, that when the given individual has no preference for one of the commodities to the exclusion of the other, their utilities are equal.

Let us see now how this definition of utility applies to the case of an individual who enters the market to purchase supplies of certain commodities *for a given period of time.* We will suppose that at the moment when we observe him he is exchanging some of a commodity A for another B. We may suppose prices to be such that he gives one gallon of A and receives two bushels of B, then a second gallon for two more bushels, and so on, until finally he has given a gallons and received b bushels. At what point does he stop? Although the *exchange values* of a gallons of A and b bushels of B are equal, their utilities to the individual considered are not. He prefers b to the exclusion of a for his act proves his preference. We write therefore,

$$\text{ut. of } b > \text{ut. of } a.$$

Why then did he cease to buy B? He sold exactly a gallons for b bushels. By stopping here he has shown that the utility of a small increment of B, say another bushel, is less to him than the utility of the corresponding number of gallons of A. But by not stopping before, he has shown that the utility of a small decrement, say a single bushel, is greater than the utility of the gallons buying it. Now, by the principle of continuity, if the small increment or decrement be diminished indefinitely in size, the two inequalities just given approach a common limiting equation,

$$\text{ut. of } db = \text{ut. of } da,$$

db and da being here exchangeable increments.

Since the last increment db is exchanged for da at the same rate as a was exchanged for b, that is, at the market rate, we have

$$a/b = da/db,$$

each ratio being the price of a in terms of b, or as the equation may also be written,

$$db/b = da/a.$$

Dividing the first equation by this, member by member, we obtain

$$b\frac{\text{ut. of } db}{db} = a\frac{\text{ut. of } da}{da}.$$

No confusion will arise if we write this last equation in the form

$$b\frac{dU}{db} = a\frac{dU}{da}.$$

The differential coefficients here introduced are known as *marginal utilities. The marginal utility of a commodity is to be defined as the utility of an infinitesimal increment of that commodity, on the margin of a given rate of consumption, divided by the amount of that increment;* or, more exactly, it is the *limit of the ratio* thus found. It will vary with the rate of consumption, that is, with the amount of the commodity available during a given period of time to satisfy the necessities or desires of the individual considered. *For infinitesimal increments on the margin of the same amount of the commodity it may be assumed that the utility is directly proportional to the size of the increment.* It is plain, then, that the ratio which determines marginal utility does not vary with the absolute value of the infinitesimal increment. The equation deduced above may be expressed thus: *For a given purchaser at the time of purchase the quantity of the commodity purchased multiplied by its marginal utility is equal to the like product for the commodity sold.*

The marginal utility of the two commodities will appear from the preceding theorem to be in the same ratio as their prices. This is true for any given instant, but not in general. If we assume that one of the two commodities exchanged is money, the amount paid being m, we have $a\,dU/da = m\,dU/dm$, where dU/dm is the marginal utility of money. In the case of a given individual this may vary in value in the same way as the marginal utility of any other commodity, and with different individuals in the same circumstances it may have altogether different values.

We wish now to compare two marginal utilities with one another. These marginal utilities may relate to different commodities, or they may relate to the same commodity on the margin of different rates of consumption. In either case we select a new commodity, which must be supposed, of course, to be uninfluenced as to marginal utility by the variations which may be supposed in connection with the given commodity or commodities. We take two *equal* infinitesimal increments of the given commodity or commodities: they

will have utilities proportional to the marginal utilities in question. Having assumed now any convenient rate of consumption for the new commodity, compare infinitesimal increments of it which have respectively the *same utilities* as the increments previously considered. Infinitesimal increments of the same commodity on the margin of the same amount have utilities proportional to their magnitudes. The ratio of these increments is, then, the same as the ratio of the infinitesimal utilities of the original commodity or commodities, and consequently the same as that of the marginal utilities.

In order now to establish a unit of utility we must select some standard commodity, some standard rate of consumption, and some standard amount of the commodity which is to possess the unit utility. For example, to the man who consumes 200 loaves of bread a year an additional loaf possesses a utility which might be adopted as a unit. This is all, however, under the assumption, as is also all our preceding discussion, that the utility of bread, and of every other commodity, is independent of that of every other commodity. Dr. Fisher introduces the term *util*, which he defines in the following manner: *A util is the marginal utility of any arbitrarily chosen commodity on the margin of some arbitrarily chosen quantity of that commodity for a given individual at a given time.*

To get the *total utility* of a given amount of a given commodity we summate the utilities of its several parts. For example, the utility of 100 loaves of bread is the sum of the utilities of each loaf.

$$\text{ut. of } x = \text{ut. } (dx_1) + \text{ut. } (dx_2) + \ldots + \text{ut. } (dx_n)$$

$$= \int_0^x \text{ut. } (dx)$$

$$= \int_0^x \frac{dU}{dx}\, dx.$$

The total utility of a given quantity of a commodity at a given time and for a given individual is the integral of the marginal utility times the differential of that commodity. To illustrate this, let us suppose a curve constructed, the abscissa of each point of which indicates the amount of the commodity consumed by an individual during some stated period of time, and the ordinate the corresponding marginal utility of the commodity. Then the total utility of the amount of the commodity included between any two given values of the abscissa will be the area included between the correspond-

ing ordinates; for each infinitesimal increment of commodity will have for its utility the area vertically over it. This curve, which we will call the marginal utility curve, or Jevons's utility curve, affords us a graphical representation of these most important quantitative economic conceptions.

There are two other important quantities which have simple graphical representations in connection with Jevons's utility curve. These are *utility value* and *consumer's rent.* The utility value of a commodity is the product of the amount of the commodity by its marginal utility,

$$x\frac{dU}{dx}.$$

The name is suggested by money-value, which is quantity times price. It is represented by the rectangle contained by the abscissa and ordinate of the corresponding point of the curve.

The consumer's rent is total utility minus utility-value

$$\int_0^x \frac{dU}{dx}\,dx - x\frac{dU}{dx}.$$

It is the total utility diminished by that utility which the commodity would have if it were all rated at the same degree of utility as the last or the least useful increment. In the figure it is represented by the triangular area limited by the curve and situated above the rectangle denoting utility-value.

The last economic quantity which we shall at present introduce is *price.* In an exchange between a commodity and money we have, as we have already seen, the equation $a\frac{dU}{da} = m\frac{dU}{dm}$, a denoting the amount of commodity for which an amount of money m is paid. The price of the commodity is then

$$p_a = \frac{m}{a} = \frac{\frac{dU}{da}}{\frac{dU}{dm}},$$

or the ratio of the marginal utility of the commodity to that of money.

From an obvious extension of the last equation we may draw the following conclusion: *In an ideal market at the instant that any individual discontinues trading, the mar-*

ginal utilities to him of all the commodities are in proportion to their prices.

We may now write down the so-called *equations of exchange.* Considering the special case in which the market contains n consumers and m commodities, the amounts of the commodities $K_a, K_b, \ldots K_m$, and the incomes of the consumers $K_1, K_2, \ldots K_n$, all being given, we find the following equations for the price and marginal utility of each commodity, and for the amount purchased by each consumer.

$$\begin{cases} a_1 + a_2 + a_3 + \ldots + a_n = K_a, \\ b_1 + b_2 + b_3 + \ldots + b_n = K_b, \\ \ldots\ldots\ldots\ldots\ldots \\ m_1 + m_2 + m_3 + \ldots + m_n = K_m. \end{cases}$$

$$\begin{cases} a_1 p_a + b_1 p_b + \ldots + m_1 p_m = K_1, \\ a_2 p_a + b_2 p_b + \ldots + m_2 p_m = K_2, \\ \ldots\ldots\ldots\ldots\ldots \\ a_n p_a + b_n p_b + \ldots + m_n p_m = K_n. \end{cases}$$

$$\begin{cases} \dfrac{\dfrac{dU}{da_1}}{p_a} = \dfrac{\dfrac{dU}{db_1}}{p_b} = \ldots = \dfrac{\dfrac{dU}{dm_1}}{p_m}, \\ \dfrac{\dfrac{dU}{da_2}}{p_a} = \dfrac{\dfrac{dU}{db_2}}{p_b} = \ldots = \dfrac{\dfrac{dU}{dm_2}}{p_m}, \\ \ldots\ldots\ldots\ldots\ldots \\ \dfrac{\dfrac{dU}{da_n}}{p_a} = \dfrac{\dfrac{dU}{db_n}}{p_b} = \ldots = \dfrac{\dfrac{dU}{dm_n}}{p_m}. \end{cases}$$

$$\begin{cases} \dfrac{dU}{da_1} = f_{a_1}(a_1), \quad \dfrac{dU}{db_1} = f_{b_1}(b_1), \ldots \dfrac{dU}{d_{m_1}} = f_{m_1}(m_1), \\ \dfrac{dU}{da_2} = f_{a_2}(a_2), \quad \dfrac{dU}{db_2} = f_{b_2}(b_2), \ldots \dfrac{dU}{dm_2} = f_{m_2}(m_2), \\ \ldots\ldots\ldots\ldots\ldots \\ \dfrac{dU}{da_n} = f_{an}(a_n), \quad \dfrac{dU}{db_n} = f_{bn}(b_n), \ldots \dfrac{dU}{dm_n} = f_{mn}(m_n). \end{cases}$$

These equations are equal in number to $2mn + m$, the number of unknown quantities, and will suffice to determine them if we know, as we must assume, the nature of the functions f. Similar equations can be constructed represent-

ing the general case in which producers as well as consumers are considered..

The preceding ideas are developed with much skill in Dr. Fisher's paper. Its most conspicuous feature, however, consists in the systematic representation of different questions in the equilibrium of supply and demand through the agency of an elaborate mechanism in the construction of which the greatest ingenuity is displayed. The equilibrium is brought about by means of a liquid in which float a number of cisterns representing the individual consumers and producers. These are made to fulfil the requisite conditions and relations through a series of connecting levers. This dynamical solution of economic problems is both novel and instructive. When more general considerations are introduced, however, under which the utilities of the various commodities are supposed to influence one another, this mechanical representation becomes impossible.

In the general case of mutually dependent commodities, there is nothing to correspond to the notion of total utility, although that of marginal utility is preserved. The idea of marginal utility is thus seen to be of a more fundamental character than that of total utility. In the case where two commodities alone are considered, a convenient method of investigation is by means of two families of curves orthogonal to each other. These are called *indifference curves* and *lines of force*, or curves having at every point the direction of maximum increase of utility. The position of equilibrium for any individual is found to be the point at which his *income line* is cut orthogonally by a line of force.

Dr. Fisher does not refer to one of the most important applications of the mathematical methods, viz., the theory of monopoly revenue and taxation. This theory presents several points of peculiar interest, both theoretical and practical. We must refer the reader for an account of it to Marshall's treatise, already mentioned.

THOMAS S. FISKE.

NEW YORK, *May* 20, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, May 6, at half-past three o'clock, the president, Dr. McClintock, in the chair. Professor E. S. Crawley read a paper entitled "The forms of uninodal quartics with four real and distinct asymptotes." Taking the equation in the form $a_1w_4 + a_2w_3 + a_3w_2 = 0$, where a_1, a_2, a_3 are real constants, and w_4, w_3, w_2 are homogeneous functions of x and y of the degrees indicated by the subscripts, the relations to each other of the roots of the equations $w_4 = 0$, $w_3 = 0$, $w_2 = 0$, were examined and discussed. Professor Crawley exhibited an elaborate series of drawings, showing the various forms of the curves. A paper by Professor E. A. Engler, entitled "Geometrical constructions for cutting from a cone of revolution plane sections (*a*) of given excentricity, (*b*) of given latus rectum," was also presented to the Society. T. S. F.

THE annual meeting of the *Deutsche Mathematiker-Vereinigung* will take place at Munich, September 4–10. The exhibition of mathematical models, instruments, etc., prepared for this meeting will be open to visitors from September 1 to September 30. A. Z.

HARVARD UNIVERSITY.—Besides regular undergraduate courses, the class-room work in which will average nineteen hours a week throughout the year, the following more advanced mathematical courses are offered for the year 1893–94:—By Professor J. M. Peirce: Quaternions (first course); Algebraic plane curves; Qualitative algebra. By Professor C. J. White: Differential and integral calculus (second course); Theory of equations. By Professor Byerly: Analytic mechanics; Trigonometric series, and spherical harmonics. By Professor B. O. Peirce: Potential function; Hydrostatics and hydrokinematics. By Dr. Osgood: Higher algebra; Theory of substitutions. By Dr. Bôcher: Modern geometry; Theory of functions (first course); Functions defined by differential equations.

Each of the above courses, with a few exceptions, extends throughout the year and consists of three lectures a week. There are also offered two courses in which the work will consist of investigation and reading to be carried on by the students. These are:—By Professor J. M. Peirce: Studies in algebraic curves, quaternions, or the theory of functions, at the option of each student. By Dr. Osgood: Theory of invariants.

Attention should also be called to several courses, largely mathematical, in the departments of physics and engineering.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENNETT (G. T.). On the residues of powers of numbers for any composite modulus. London, 1893. 4to. 148 pp. 5s. 6*d*.

BERGMANN (F.). Aufgaben zur stereographischen Kugelprojection. Olmütz, 1892. 8vo. 16 pp. Mk. 1.20

BLAKE (E. M.). The method of indeterminate coefficients and exponents applied to differential equations. Dissertation submitted in partial fulfilment of the requirements for the degree of doctor of philosophy in the university faculty of pure science, Columbia College. New York, Drummond, 1893. 8vo. 44 pp.

GALLASCH (H.). Die Grundlagen der Algebra im Kant'schen Sinne. Wien, 1893. 8vo. 23 pp. Mk. 1.20

GLASER (R.). Über die Minimalflächen. Tübingen, 1891. 8vo. 42 pp. Mk. 1.20

GRÜNFELD (E.). Zur Theorie der Systeme linearer Differentialgleichungen 1ter Ordnung und der Fuchs'schen Differentialgleichungen *m*ter Ordnung. Wien, 1892. 8vo. 30 pp. Mk. 1.20

HABART (K.). Charakter und Darstellung von Wurfcurven constanter Wurfkraftrichtung. Über Systeme von Kegelschnitten mit unendlich fernem Doppelcontacte. Elbogen, 1892. 8vo. 35 pp. Mk. 1.50

KNOCHE (G.). Über die aus der complexen Multiplication der elliptischen Functionen entspringenden algebraischen Gleichungen. Marburg, 1892. 8vo. 38 pp. Mk. 1.20

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A COMPARATIVE REVIEW OF RECENT RESEARCHES IN GEOMETRY.*

(PROGRAMME ON ENTERING THE PHILOSOPHICAL FACULTY AND THE SENATE OF THE UNIVERSITY OF ERLANGEN IN 1872.)

BY PROF. FELIX KLEIN.

Prefatory Note by the Author.—My 1872 Programme, appearing as a separate publication (Erlangen, A. Deichert), had but a limited circulation at first. With this I could be satisfied more easily, as the views developed in the Programme could not be expected at first to receive much attention. But now that the general development of mathematics has taken, in the meanwhile, the direction corresponding precisely to these views, and particularly since *Lie* has begun the publication in extended form of his *Theorie der Transformationsgruppen* (Leipzig, Teubner, vol. I. 1888, vol. II. 1890), it seems proper to give a wider circulation to the expositions in my Programme. An Italian translation by M. Gino Fano was recently published in the *Annali di Matematica*, ser. 2, vol. 17. A kind reception for the English translation, for which I am much indebted to Mr. Haskell, is likewise desired.

The translation is an absolutely literal one; in the two or three places where a few words are changed, the new phrases are enclosed in square brackets [ ]. In the same way are indicated a number of additional footnotes which it seemed desirable to append, most of them having already appeared in the Italian translation.—F. KLEIN.

Among the advances of the last fifty years in the field of geometry, the development of *projective geometry* † occupies the first place. Although it seemed at first as if the so-called metrical relations were not accessible to this treatment, as they do not remain unchanged by projection, we have nevertheless learned recently to regard them also from the projective point of view, so that the projective method now embraces the whole of geometry. But metrical properties are then to be regarded no longer as characteristics of the geometrical figures *per se*, but as their relations to a fundamental configuration, the imaginary circle at infinity common to all spheres.

When we compare the conception of geometrical figures gradually obtained in this way with the notions of ordinary (elementary) geometry, we are led to look for a general principle in accordance with which the development of both methods has been possible. This question seems the more important as, beside the elementary and the projective geometry, are arrayed a series of other methods, which, albeit they are less developed, must be allowed the same right to an individual existence. Such are the geometry of reciprocal radii

* Translated by Dr. M. W. HASKELL, Assistant Professor of Mathematics in the University of California.

† See Note I. of the appendix.

vectores, the geometry of rational transformations, etc., which will be mentioned and described further on.

In undertaking in the following pages to establish such a principle, we shall hardly develop an essentially new idea, but rather formulate clearly what has already been more or less definitely conceived by many others. But it has seemed the more justifiable to publish connective observations of this kind, because geometry, which is after all one in substance, has been only too much broken up in the course of its recent rapid development into a series of almost distinct theories,* which are advancing in comparative independence of each other. At the same time I was influenced especially by a wish to present certain methods and views that have been developed in recent investigations by *Lie* and myself. Our respective investigations, different as has been the nature of the subjects treated, have led to the same generalized conception here presented; so that it has become a sort of necessity to thoroughly discuss this view and on this basis to characterize the contents and general scope of those investigations.

Though we have spoken so far only of geometrical investigations, we will include investigations on manifoldnesses of any number of dimensions,† which have been developed from geometry by making abstraction from the geometric image, which is not essential for purely mathematical investigations.‡ In the investigation of manifoldnesses the same different types occur as in geometry; and, as in geometry, the problem is to bring out what is common and what is distinctive in investigations undertaken independently of each other. Abstractly speaking, it would in what follows be sufficient to speak throughout of manifoldnesses of n dimensions simply; but it will render the exposition simpler and more intelligible to make use of the more familiar space-perceptions. In proceeding from the consideration of geometric objects and developing the general ideas by using these as an example, we follow the path which our science has taken in its development and which it is generally best to pursue in its presentation.

A preliminary exposition of the contents of the following pages is here scarcely possible, as it can hardly be presented in a more concise form;§ the headings of the sections will indicate the general course of thought.

At the end I have added a series of notes, in which I have

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* See Note II. † See Note IV. ‡ See Note III.

§ This very conciseness is a defect in the following presentation which I fear will render the understanding of it essentially more difficult. But the difficulty could hardly be removed except by a very much fuller exposition, in which the separate theories, here only touched upon, would have been developed at length.

either developed further single points, wherever the general exposition of the text would seem to demand it, or have tried to define with reference to related points of view the abstract mathematical one predominant in the observations of the text.

§ 1.

GROUPS OF SPACE-TRANSFORMATIONS. PRINCIPAL GROUP. FORMULATION OF A GENERAL PROBLEM.

The most essential idea required in the following discussion is that of a *group* of space-transformations.

The combination of any number of transformations of space * is always equivalent to a single transformation. If now a given system of transformations has the property that any transformation obtained by combining any transformations of the system belongs to that system, it shall be called a *group of transformations.*†

An example of a group of transformations is afforded by the totality of motions, every motion being regarded as an operation performed on the whole of space. A group contained in this group is formed, say, by the rotations about one point.‡ On the other hand, a group containing the group of motions is presented by the totality of the collineations. But the totality of the dualistic transformations does not form a group; for the combination of two dualistic transformations is equivalent to a collineation. A group is, however, formed by adding the totality of the dualistic to that of the collinear transformations. §

* We always regard the totality of configurations in space as simultaneously affected by the transformations, and speak therefore of *transformations of space.* The transformations may introduce other elements in place of points, like dualistic transformations, for instance; there is no distinction in the text in this regard.

† [This definition is not quite complete, for it has been tacitly assumed that the groups mentioned always include the inverse of every operation they contain; but, when the number of operations is infinite, this is by no means a necessary consequence of the group idea, and this assumption of ours should therefore be explicitly added to the definition of this idea given in the text.]

The ideas, as well as the notation, are taken from the *theory of substitutions,* with the difference merely that there instead of the transformations of a continuous region the permutations of a finite number of discrete quantities are considered.

‡ *Camille Jordan* has formed all the groups contained in the general group of motions: *Sur les groupes de mouvements,* Annali di Matematica, vol. 2.

§ It is not at all necessary for the transformations of a group to form a continuous succession, although the groups to be mentioned in the text will indeed always have that property. For example, a group is formed by the finite series of motions which superpose a regular body upon itself, or by the infinite but discrete series which superpose a sine-curve upon itself.

Now there are space-transformations by which the geometric properties of configurations in space remain entirely unchanged. For geometric properties are, from their very idea, independent of the position occupied in space by the configuration in question, of its absolute magnitude, and finally of the sense * in which its parts are arranged. The properties of a configuration remain therefore unchanged by any motions of space, by transformation into similar configurations, by transformation into symmetrical configurations with regard to a plane (reflection), as well as by any combination of these transformations. The totality of all these transformations we designate as the *principal group* † of space-transformations; *geometric properties are not changed by the transformations of the principal group.* And, conversely, *geometric properties are characterized by their remaining invariant under the transformations of the principal group.* For, if we regard space for the moment as immovable, etc., as a rigid manifoldness, then every figure has an individual character; of all the properties possessed by it as an individual, only the properly geometric ones are preserved in the transformations of the principal group. The idea, here formulated somewhat indefinitely, will be brought out more clearly in the course of the exposition.

Let us now dispense with the concrete conception of space, which for the mathematician is not essential, and regard it only as a manifoldness of n dimensions, that is to say, of three dimensions, if we hold to the usual idea of the point as space element. By analogy with the transformations of space we speak of transformations of the manifoldness; they also form groups. But there is no longer, as there is in space, one group distinguished above the rest by its signification; each group is of equal importance with every other. As a generalization of geometry arises then the following comprehensive problem:

Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.

To make use of a modern form of expression, which to be sure is ordinarily used only with reference to a particular group, the group of all the linear transformations, the problem might be stated as follows:

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* By "sense" is to be understood that peculiarity of the arrangement of the parts of a figure which distinguishes it from the symmetrical figure (the reflected image). Thus, for example, a right-handed and a left-handed helix are of opposite "sense."

† The fact that these transformations form a group results from their very idea.

Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group.

This is the general problem, and it comprehends not alone ordinary geometry, but also and in particular the more recent geometrical theories which we propose to discuss, and the different methods of treating manifoldnesses of n dimensions. Particular stress is to be laid upon the fact that the choice of the group of transformations to be adjoined is quite arbitrary, and that consequently all the methods of treatment satisfying our general condition are in this sense of equal value.

§ 2.

Groups of Transformations, one of which includes the other, are successively adjoined. The Different Types of Geometrical Investigation and their Relation to each other.

As the geometrical properties of configurations in space remain unaltered under *all* the transformations of the principal group, it is by the nature of the question absurd to inquire for such properties as would remain unaltered under only a part of those transformations. This inquiry becomes justified, however, as soon as we investigate the configurations of space in their relation to elements regarded as fixed. Let us, for instance, consider the configurations of space with reference to one particular point, as in spherical trigonometry. The problem then is to develop the properties remaining invariant under the transformations of the principal group, not for the configurations taken independently, but for the system consisting of these configurations together with the given point. But we can state this problem in this other form: to examine configurations in space with regard to such properties as remain unchanged by those transformations of the principal group which can still take place when the point is kept fixed. In other words, it is exactly the same thing whether we investigate the configurations of space taken in connection with the given point from the point of view of the principal group or whether, without any such connection, we replace the principal group by that partial group whose transformations leave the point in question unchanged.

This is a principle which we shall frequently apply; we will therefore at once formulate it generally, as follows:

Given a manifoldness and a group of transformations applying to it. Let it be proposed to examine the configurations contained in the manifoldness with reference to a given configuration. *We may, then, either add the given configuration*

to the system, and then we have to investigate the properties of the extended system from the point of view of the given group, or we may leave the system unextended, limiting the transformations to be employed to such transformations of the given group as leave the given configuration unchanged. (These transformations necessarily form a group by themselves.)

Let us now consider the converse of the problem proposed at the beginning of this section. This is intelligible from the outset. We inquire what properties of the configurations of space remain unaltered by a group of transformations which contains the principal group as a part of itself. Every property found by an investigation of this kind is a geometric property of the configuration itself; but the converse is not true. In the converse problem we must apply the principle just enunciated, the principal group being now the smaller. We have then:

If the principal group be replaced by a more comprehensive group, a part only of the geometric properties remain unchanged. The remainder no longer appear as properties of the configurations of space by themselves, but as properties of the system formed by adding to them some particular configuration. This latter is defined, in so far as it is a definite * configuration at all, by the following condition: *The assumption that it is fixed must restrict us to those transformations of the given group which belong to the principal group.*

In this theorem is to be found the peculiarity of the recent geometrical methods to be discussed here, and their relation to the elementary method. What characterizes them is just this, that they base their investigations upon an extended group of space-transformations instead of upon the principal group. Their relation to each other is defined, when one of the groups includes the other, by a corresponding theorem. The same is true of the various methods of treating manifoldnesses of n dimensions which we shall take up. We shall now consider the separate methods from this point of view, and this will afford an opportunity to explain on concrete examples the theorems enunciated in a general form in this and the preceding sections.

§ 3.

Projective Geometry.

Every space-transformation not belonging to the principal group can be used to transfer the properties of known config-

* Such a configuration can be generated, for instance, by applying the transformations of the principal group to any arbitrary element which cannot be converted into itself by any transformation of the given group.

urations to new ones. Thus we apply the results of plane geometry to the geometry of surfaces that can be represented (*abgebildet*) upon a plane; in this way long before the origin of a true projective geometry the properties of figures derived by projection from a given figure were inferred from those of the given figure. But projective geometry only arose as it became customary to regard the original figure as essentially identical with all those deducible from it by projection, and to enunciate the properties transferred in the process of projection in such a way as to put in evidence their independence of the change due to the projection. By this process *the group of all the projective transformations* was made the basis of the theory in the sense of § 1, and that is just what created the antithesis between projective and ordinary geometry.

A course of development similar to the one here described can be regarded as possible in the case of every kind of space-transformation; we shall often refer to it again. It has gone on still further in two directions within the domain of projective geometry itself. On the one hand, the conception was broadened by admitting the *dualistic* transformations into the group of the fundamental transformations. From the modern point of view two reciprocal figures are not to be regarded as two distinct figures, but as essentially one and the same. A further advance consisted in extending the fundamental group of collinear and dualistic transformations by the admission in each case of the *imaginary* transformations. This step requires that the field of true space-elements has previously been extended so as to include imaginary elements, —just exactly as the admission of dualistic transformations into the fundamental group requires the simultaneous introduction of point and line as space-elements. This is not the place to point out the utility of introducing imaginary elements, by means of which alone we can attain an exact correspondence of the theory of space with the established system of algebraic operations. But, on the other hand, it must be remembered that the reason for introducing the imaginary elements is to be found in the consideration of algebraic operations and not in the group of projective and dualistic transformations. For, just as we can in the latter case limit ourselves to real transformations, since the real collineations and dualistic transformations form a group by themselves, so we can equally well introduce imaginary space-elements even when we are not employing the projective point of view, and indeed must do so in strictly algebraic investigations.

How metrical properties are to be regarded from the projective point of view is determined by the general theorem of the preceding section. Metrical properties are to be considered as projective relations to a fundamental configuration,

the circle at infinity,* a configuration having the property that it is transformed into itself only by those transformations of the projective group which belong at the same time to the principal group. The proposition thus broadly stated needs a material modification owing to the limitation of the ordinary view taken of geometry as treating only of *real* space-elements (and allowing only *real* transformations). In order to conform to this point of view, it is necessary expressly to adjoin to the circle at infinity the system of real space-elements (points); properties in the sense of elementary geometry are projectively either properties of the configurations by themselves, or relations to this system of the real elements, or to the circle at infinity, or finally to both.

We might here make mention further of the way in which *von Staudt* in his "Geometrie der Lage" (Nürnberg, 1847) develops projective geometry,—i.e., that projective geometry which is based on the group containing all the real projective and dualistic transformations.†

We know how, in his system, he selects from the ordinary matter of geometry only such features as are preserved in projective transformations. Should we desire to proceed to the consideration of metrical properties also, what we should have to do would be precisely to introduce these latter as relations to the circle at infinity. The course of thought thus brought to completion is in so far of great importance for the present considerations, as a corresponding development of geometry is possible for every one of the methods we shall take up.

§ 4.

Transfer of Properties by Representation (Abbildung).

Before going further in the discussion of the geometrical methods which present themselves beside the elementary and the projective geometry, let us develop in a general form certain considerations which will continually recur in the course of the work, and for which a sufficient number of examples are already furnished by the subjects touched upon up to this point. The present section and the following one will be devoted to these discussions.

* This view is to be regarded as one of the most brilliant achievements of [the French school]; for it is precisely what provides a sound foundation for that distinction between properties of position and metrical properties, which furnishes a most desirable starting-point for projective geometry.

† The extended horizon, which includes *imaginary* transformations, was first used by *von Staudt* as the basis of his investigations in his later work, "Beiträge zur Geometrie der Lage" (Nürnberg, 1856–60).

Suppose a manifoldness A has been investigated with reference to a group B. If, by any transformation whatever, A be then converted into a second manifoldness A', the group B of transformations, which transformed A into itself, will become a group B', whose transformations are performed upon A'. It is then a self-evident principle that *the method of treating A with reference to B at once furnishes the method of treating A' with reference to B'*, i.e., every property of a configuration contained in A obtained by means of the group B furnishes a property of the corresponding configuration in A' to be obtained by the group B'.

For example, let A be a straight line and B the ∞^3 linear transformations which transform A into itself. The method of treating A is then just what modern algebra designates as the theory of binary forms. Now, we can establish a correspondence between the straight line and a conic section A' in the same plane by projection from a point of the latter. The linear transformations B of the straight line into itself will then become, as can easily be shown, the linear transformations B' of the conic into itself, i.e., the changes of the conic resulting from those linear transformations of the plane which transform the conic into itself.

Now, by the principle stated in § 2,* the study of the geometry of the conic section is the same, whether the conic be regarded as fixed and only those linear transformations of the plane which transform the conic into itself be taken into account, or whether all the linear transformations of the plane be considered and the conic be allowed to vary too. The properties which we recognized in systems of points on the conic are accordingly projective properties in the ordinary sense. Combining this consideration with the result just deduced, we have, then:

The theory of binary forms and the projective geometry of systems of points on a conic are one and the same, i.e., to every proposition concerning binary forms corresponds a proposition concerning such systems of points, and vice versa.†

Another suitable example to illustrate these considerations is the following. If a quadric surface be brought into correspondence with a plane by stereographic projection, the surface will have one fundamental point,—the centre of projection. In the plane there are two,—the projections of the generators passing through the centre of projection. It then follows directly: the linear transformations of the plane which

* The principle might be said to be applied here in a somewhat extended form.

† Instead of the plane conic we may equally well introduce a twisted cubic, or indeed a corresponding configuration in an n-dimensional manifoldness.

leave the two fundamental points unaltered are converted by the representation (*Abbildung*) into linear transformations of the quadric into itself, but only into those which leave the centre of projection unaltered. By linear transformations of the surface into itself are here meant the changes undergone by the surface when linear space-transformations are performed which transform the surface into itself. According to this, the projective investigation of a plane with reference to two of its points is identical with the projective investigation of a quadric surface with reference to one of its points. Now, if imaginary elements are also taken into account, the former is nothing else but the investigation of the plane from the point of view of elementary geometry. For the principal group of plane transformations comprises precisely those linear transformations which leave two points (the circular points at infinity) unchanged. We obtain then finally:

Elementary plane geometry and the projective investigation of a quadric surface with reference to one of its points are one and the same.

These examples may be multiplied at pleasure;* the two here developed were chosen because we shall have occasion to refer to them again.

§ 5.

On the Arbitrariness in the Choice of the Space-Element. Hesse's Principle of Transference. Line Geometry.

As element of the straight line, of the plane, of space, or of any manifoldness to be investigated, we may use instead of the point any configuration contained in the manifoldness,—a group of points, a curve or surface,† etc. As there is nothing at all determined at the outset about the number of arbitrary parameters upon which these configurations shall depend, the number of dimensions of our line, plane, space, etc., may be anything we like, according to our choice of the element. *But as long as we base our geometrical investigation upon the same group of transformations, the substance of the geometry remains unchanged.* That is to say, every proposition resulting from *one* choice of the space-element will be a true proposition under any other assumption; but the

* For other examples, and particularly for the extension to higher dimensions of which those here presented are capable, let me refer to an article of mine: *Ueber Liniengeometrie und metrische Geometrie* (Mathematische Annalen, vol. 5), and further to *Lie*'s investigations cited later.

† See Note III.

arrangement and correlation of the propositions will be changed.

The essential thing is, then, the group of transformations; the number of dimensions to be assigned to a manifoldness appears of secondary importance.

The combination of this remark with the principle of the last section furnishes many interesting applications, some of which we will now develop, as these examples seem better fitted to explain the meaning of the general theory than any lengthy exposition.

Projective geometry on the straight line (the theory of binary forms) is, by the last section, equivalent to projective geometry on the conic. Let us now regard as element on the conic the point-pair instead of the point. Now, the totality of the point-pairs of the conic may be brought into correspondence with the totality of the straight lines in the plane, by letting every line correspond to that point-pair in which it intersects the conic. By this representation (*Abbildung*) the linear transformations of the conic into itself are converted into those linear transformations of the plane (regarded as made up of straight lines) which leave the conic unaltered. But whether we consider the group of the latter, or whether we base our investigation on the totality of the linear transformations of the plane, always adjoining the conic to the plane configurations under investigation, is by § 2 one and the same thing. Uniting all these considerations, we have:

The theory of binary forms and projective geometry of the plane with reference to a conic are identical.

Finally, as projective geometry of the plane with reference to a conic, by reason of the equality of its group, coincides with that projective metrical geometry which in the plane can be based upon a conic, * we can also say:

The theory of binary forms and general projective metrical geometry in the plane are one and the same.

In the preceding consideration the conic in the plane might be replaced by the twisted cubic, etc., but we will not carry this out further. The correlation here explained between the geometry of the plane, of space, or of a manifoldness of any number of dimensions is essentially identical with the principle of transference proposed by *Hesse* (Borchardt's Journal, vol. 66).

An example of much the same kind is furnished by the projective geometry of space; or, in other words, the theory of quaternary forms. If the straight line be taken as space-element and be determined, as in line geometry, by six homogeneous co-ordinates connected by a quadratic equation of

* See Note V.

condition, the linear and dualistic transformations of space are seen to be those linear transformations of the six variables (regarded as independent) which transform the equation of condition into itself. By a combination of considerations similar to those just developed, we obtain the following theorem :

The theory of quaternary forms is equivalent to projective measurement in a manifoldness generated by six homogeneous variables.

For a detailed exposition of this view I will refer to an article in the Math. Annalen (vol. 6): "Ueber die sogenannte Nicht-Euklidische Geometrie" [Zweiter Aufsatz], and to a note at the close of this paper.*

To the foregoing expositions I will append two remarks, the first of which is to be sure implicitly contained in what has already been said, but needs to be brought out at length, because the subject to which it applies is only too likely to be misunderstood.

Through the introduction of arbitrary configurations as space-elements, space becomes of any number of dimensions we like. But if we then keep to the (elementary or projective) space-perception with which we are familiar, the fundamental group for the manifoldness of n dimensions is given at the outset ; in the one case it is the principal group, in the other the group of projective transformations. If we wished to take a different group as a basis, we should have to depart from the ordinary (or from the projective) space-perception. Thus, while it is correct to say that, with a proper choice of space-elements, space represents manifoldnesses of any number of dimensions, it is equally important to add that *in this representation either a definite group must form the basis of the investigation of the manifoldness, or else, if we wish to choose the group, we must broaden our geometrical perception accordingly.* If this were overlooked, an interpretation of line geometry, for instance, might be sought in the following way. In line geometry the straight line has six co-ordinates; the conic in the plane has the same number of coefficients. The interpretation of line geometry would then be the geometry in a system of conics separated from the aggregation of all conics by a quadratic equation between the coefficients. This is correct, provided we take as fundamental group for plane geometry the totality of the transformations represented by the linear transformations of the coefficients of the conic which transform the quadratic equation into itself. But if we retain the elementary or the projective view of plane geometry, we have no interpretation at all.

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* See Note VI.

The second remark has reference to the following line of reasoning: Suppose in space some group or other, the principal group for instance, be given. Let us then select a single configuration, say a point, or a straight line, or even an ellipsoid, etc., and apply to it all the transformations of the principal group. We thus obtain an infinite manifoldness with a number of dimensions in general equal to the number of arbitrary parameters contained in the group, but reducing in special cases, namely, when the configuration originally selected has the property of being transformed into itself by an infinite number of the transformations of the group. Every manifoldness generated in this way may be called, with reference to the generating group, a *body*.*

If now we desire to base our investigations upon the group, selecting at the same time certain definite configurations as space-elements, and if we wish to represent uniformly things which are of like characteristics, *we must evidently choose our space-elements in such a way that their manifoldness either is itself a body or can be decomposed into bodies.* This remark, whose correctness is evident, will find application later (§ 9). This idea of a body will come under discussion once more in the closing section, in connection with certain related ideas.†

§ 6.

The Geometry of Reciprocal Radii. Interpretation of $x + iy$.

With this section we return to the discussion of the various lines of geometric research, which was begun in §§ 2 and 3.

As a parallel in many respects to the processes of projective geometry, we may consider a class of geometric investigations in which the transformation by reciprocal radii vectores (geometric inversion) is continually employed. To these belong investigations on the so-called cyclides and other anallagmatic surfaces, on the general theory of orthogonal systems, likewise on potential, etc. It is true that the processes here involved

* In choosing this name I follow the precedent established by *Dedekind* in the theory of numbers, where he applies the name *body* to a system of numbers formed from given elements by given operations (*Dirichlet's* Vorlesungen über Zahlentheorie, 2. Aufl.).

† [In the text sufficient attention is not paid to the fact that the proposed group may contain so-called self-conjugate subgroups. If a geometrical configuration remain unchanged by the operations of a self-conjugate subgroup, the same is true for all configurations into which it is transformed by the operations of the whole group; i.e., for all configurations of the body arising from it. But a body so formed would be absolutely unsuited to represent the operations of the group. In the text, therefore, are to be admitted only bodies formed of space-elements which remain unchanged by no self-conjugate subgroup of the given group whatever.]

have not yet, like projective geometry, been united into a special geometry, *whose fundamental group would be the totality of the transformations resulting from a combination of the principal group with geometric inversion;* but this may be ascribed to the fact that the theories named have never happened to receive a connected treatment. To the individual investigators in this line of work some such systematic conception can hardly have been foreign.

The parallel between this geometry of reciprocal radii and projective geometry is apparent as soon as the question is raised; it will therefore be sufficient to call attention in a general way to the following points:

In projective geometry the elementary ideas are the point, line, and plane. The circle and the sphere are but special cases of the conic section and the quadric surface. The region at infinity of elementary geometry appears as a plane; the fundamental configuration to which elementary geometry is referred is an imaginary conic at infinity.

In the geometry of reciprocal radii the elementary ideas are the point, circle, and sphere. The line and the plane are special cases of the latter, characterized by the property that they contain a point which, however, has no further special significance in the theory, namely, the point at infinity. If we regard this point as fixed, elementary geometry is the result.

The geometry of reciprocal radii admits of being stated in a form which places it alongside of the theory of binary forms and of line geometry, provided the latter be treated in the way indicated in the last section. To this end we will for the present restrict our observations to plane geometry and therefore to the geometry of reciprocal radii in the plane.*

We have already referred to the connection between elementary plane geometry and the projective geometry of the quadric surface with one distinctive point (§ 4). If we disregard the distinctive point, that is to say, if we consider the projective geometry on the surface by itself, we have a representation of the geometry of reciprocal radii in the plane. For it is easy to see † that to the group of geometric inversion in the plane corresponds by virtue of the representation (*Abbildung*) of the quadric surface the totality of the linear transformations of the latter into itself. We have, therefore,

The geometry of reciprocal radii in the plane and the pro-

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* The geometry of reciprocal radii on the straight line is equivalent to the projective investigation of the line, as the transformations in question are the same. Thus in the geometry of reciprocal radii, also, we can speak of the anharmonic ratio of four points on a line and of four points on a circle.

† See the article already cited: *Ueber Liniengeometrie und metrische Geometrie*, Mathematische Annalen, vol. 5.

jective geometry on a quadric surface are one and the same; and, similarly:

The geometry of reciprocal radii in space is equivalent to the projective treatment of a manifoldness represented by a quadratic equation between five homogeneous variables.

By means of the geometry of reciprocal radii space geometry is thus brought into exactly the same connection with a manifoldness of four dimensions as by means of [projective] geometry with a manifoldness of five dimensions.

The geometry of reciprocal radii in the plane, if we limit ourselves to *real* transformations, admits of an interesting interpretation, or application, in still another direction. For, representing the complex variable $x + iy$ in the plane in the usual way, to its linear transformations corresponds the group of geometric inversion, with the above-mentioned restriction to real operations.* But the investigation of functions of a complex variable, regarded as subject to any linear transformations whatever, is merely what, under a somewhat different mode of representation, is called the theory of binary forms. In other words:

The theory of binary forms finds interpretation in the geometry of reciprocal radii in the real plane, and precisely in the way in which complex values of the variables are represented.

From the plane we will ascend to the quadric surface, to return to the more familiar circle of ideas of the projective transformations. As we have taken into consideration only real elements of the plane, it is not a matter of indifference how the surface is chosen; it can evidently not be a ruled surface. In particular, we may regard it as a spherical surface,—as is customary for the interpretation of a complex variable,—and obtain in this way the theorem:

The theory of the binary forms of a complex variable finds representation in the projective geometry of the real spherical surface.

I could not refrain from setting forth in a note † how admirably this interpretation illustrates the theory of binary cubics and quartics.

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* [The language of the text is inexact. To the linear transformations $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$ (where $z' = x' + iy'$, $z = x + iy$) correspond only those operations of the group of geometric inversion by which no reversion of the angles takes place (in which the two circular points of the plane are not interchanged). If we wish to include the whole group of geometric inversion, we must, in addition to the transformations mentioned, take account of the other (not less important) ones given by the formula $z' = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}$ (where again $z' = x' + iy'$, but $\bar{z} = x - iy$).]

† See Note VII.

§ 7.

Extension of the Preceding Considerations. Lie's Sphere Geometry.

With the theory of binary forms, the geometry of reciprocal radii, and line geometry, which in the foregoing pages appear co-ordinated and only distinguished by the number of variables, may be connected certain further developments, which shall now be explained. In the first place, these developments are intended to illustrate with new examples the idea that the group determining the treatment of given subjects can be extended indefinitely; but, in the second place, the intention was particularly to explain the relation to the views here set forth of certain considerations presented by *Lie* in a recent article.* The way by which we here arrive at *Lie's* sphere geometry differs in this respect from the one pursued by *Lie*, that he proceeds from the conceptions of line geometry, while we assume a smaller number of variables in our exposition. This will enable us to be in agreement with the usual geometric perception and to preserve the connection with what precedes. The investigation is independent of the number of variables, as *Lie* himself has already pointed out (Göttinger Nachrichten, 1871, Nos. 7, 22). It belongs to that great class of investigations concerned with the projective discussion of quadratic equations between any number of variables, —investigations upon which we have already touched several times, and which will repeatedly meet us again (see § 10, for instance).

I proceed from the connection established between the real plane and the sphere by stereographic projection. In § 5 we connected plane geometry with the geometry on a conic section by making the straight line in the plane correspond to the point-pair in which it meets the conic. Similarly we can establish a connection between space geometry and the geometry on the sphere, by letting every plane of space correspond to the circle in which it cuts the sphere. If then by stereographic projection we transfer the geometry on the sphere from the latter to the plane (every circle being thereby transformed into a circle), we have the following correspondence:

the space geometry whose element is the plane and whose group is formed of the linear transformations converting a sphere into itself, and

the plane geometry whose element is the circle and whose group is the group of geometric inversion.

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* *Partielle Differentialgleichungen und Complexe*, Mathematische Annalen, vol. 5.

The former geometry we will now generalize in two directions by substituting for its group a more comprehensive group. The resulting extension may then be immediately transferred to plane geometry by representation (*Abbildung*).

Instead of those linear transformations of space (regarded as made up of planes) which convert the sphere into itself, it readily suggests itself to select either the totality of the *linear* transformations of space, or the totality of those plane-transformations which leave the sphere unchanged [in a sense yet to be explained]; in the former case we dispense with the sphere, in the latter with the linear character of the transformations. The former generalization is intelligible without further explanation; we will therefore consider it first and follow out its importance for plane geometry. To the second case we shall return later, and shall then in the first place have to determine the most general transformation of that kind.

Linear space-transformations have the common property of converting pencils and sheafs of planes into like pencils and sheafs. Now, transferred to the sphere, the pencil of planes gives a pencil of circles, i.e., a system of ∞^1 circles with common intersections; the sheaf of planes gives a sheaf of circles, i.e., a system of ∞^2 circles perpendicular to a fixed circle (the circle whose plane is the polar plane of the point common to the planes of the given sheaf). Hence to linear space-transformations there correspond on the sphere, and furthermore in the plane, circle-transformations characterized by the property that they convert pencils and sheafs of circles into the same.* *The plane geometry which employs the group of transformations thus obtained is the representation of ordinary projective space geometry.* In this geometry the point cannot be used as element of the plane, for the points do not form a *body* (§ 5) for the chosen group of transformations; but circles shall be chosen as elements.

In the case of the second extension named, the first question to be settled is with regard to the nature of the group of transformations in question. The problem is, to find plane-transformations converting every [pencil] of planes whose [axis touches] the sphere into a like [pencil]. For brevity of expression, we will first consider the reciprocal problem and, moreover, go down a step in the number of dimensions; we will therefore look for point-transformations of the plane which convert every tangent to a given conic into a like tangent. To this end we regard the plane with its conic as the representation of a quadric surface projected on the plane

* Such transformations are considered in Grassmann's *Ausdehnungslehre* (edition of 1862, p. 278).

from a point of space not in the surface in such a way that the conic in question represents the boundary curve. To the tangents to the conic correspond the generators of the surface, and the problem is reduced to that of finding the totality of the point-transformations of the surface into itself by which generators remain generators.

Now, the number of these transformations is, to be sure, ∞^n, where n may have any value. For we only need to regard the point on the surface as intersection of the generators of the two systems, and to transform each system of lines into itself in any way whatever. But among these are in particular the linear transformations, and to these alone will we attend. For, if we had to do, not with a surface, but with an n-dimensional manifoldness represented by a quadratic equation, the linear transformations alone would remain, the rest would disappear.*

These linear transformations of the surface into itself, transferred to the plane by projection (other than stereographic), give two-valued point-transformations, by which from every tangent to the boundary conic is produced, it is true, a tangent, but from every other straight line in general a conic having double contact with the boundary curve. This group of transformations will be conveniently characterized by basing a projective measurement upon the boundary conic. The transformations will then have the property of converting points whose distance apart is zero by this measurement, and also points whose distance from a given point is constant, into points having the same properties.

All these considerations may be extended to any number of variables, and can in particular be applied to the original inquiry, which had reference to the sphere and plane as elements. We can then give the result an especially perspicuous form, because the angle formed by two planes according to the projective measurement referred to a sphere is equal to the angle in the ordinary sense formed by the circles in which they intersect the sphere.

We thus obtain upon the sphere, and furthermore in the plane, a group of circle-transformations having the property that *they convert circles which are tangent to each other* (*include a zero angle*), *and also circles making equal angles with another circle, into like circles.* The group of these transformations contains on the sphere the linear transformations,

* If the manifoldness be stereographically projected, we obtain the well-known theorem: in regions of n dimensions (even in space) there are no isogonal point-transformations except the transformations of the group of geometric inversion. In the plane, on the other hand, there are any number besides. See the articles by *Lie* already cited.

in the plane the transformations of the group of geometric inversion.*

The circle geometry based on this group is analogous to the sphere geometry which *Lie* has devised for space and which appears of particular importance for investigations on the curvature of surfaces. It includes the geometry of reciprocal radii in the same sense as the latter includes elementary geometry.

The circle- (sphere-) transformations thus obtained have, in particular, the property of converting circles (spheres) which touch each other into circles (spheres) having the same property. If we regard all curves (surfaces) as envelopes of circles (spheres), then it results from this fact that curves (surfaces) which touch each other will always be transformed into curves (surfaces) having the same property. The transformations in question belong, therefore, to the class of *contact-transformations* to be considered from a general standpoint further on, i.e., transformations under which the contact of point-configurations is an invariant relation. The first circle-transformations mentioned in the present section, which find their parallel in corresponding sphere-transformations, are not contact-transformations.

While these two kinds of generalization have here been applied only to the geometry of reciprocal radii, they nevertheless hold in a similar way for line geometry and in general for the projective investigation of a manifoldness defined by a quadratic equation, as we have already indicated, but shall not develop further in this connection.

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* [Perhaps the addition of some few analytical formulæ will materially help to explain the remarks in the text. Let the equation of the sphere, which we project stereographically on the plane, be in ordinary tetrahedral co-ordinates:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

The x's satisfying this equation of condition we then interpret as tetracyclic co-ordinates in the plane.

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0$$

will be the general circular equation of the plane. If we compute the radius of the circle represented in this way, we come upon the square root $\sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$, which we may denote by $i\,u_5$. We can now regard the circles as elements of the plane. The group of geometric inversion is then represented by the totality of those homogeneous linear transformations of u_1, u_2, u_3, u_4, by which $u_1^2 + u_2^2 + u_3^2 + u_4^2$ is converted into a multiple of itself. But the extended group which corresponds to *Lie*'s sphere geometry consists of those homogeneous linear transformations of the five variables u_1, u_2, u_3, u_4, u_5, which convert $u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2$ into a multiple of itself.]

§ 8.

ENUMERATION OF OTHER METHODS BASED ON A GROUP OF POINT-TRANSFORMATIONS.

Elementary geometry, the geometry of reciprocal radii, and likewise projective geometry, if we disregard the dualistic transformations connected with the interchange of the space-element, are included as special cases among the large number of conceivable methods based on groups of point-transformations. We will here mention especially only the three following methods, which agree in this respect with those named. Though these methods are far from having been developed into independent theories in the same degree as projective geometry, yet they can clearly be traced in the more recent investigations.*

1. *The Group of Rational Transformations.*

In the case of rational transformations we must carefully distinguish whether they are rational for *all* points of the region under consideration, viz., of space, or of the plane, etc., or only for the points of a manifoldness contained in the region, viz., a surface or curve. The former alone are to be employed when the problem is to develop a geometry of space or of the plane in the meaning hitherto understood ; the latter obtain a meaning, from our point of view, only when we wish to study the geometry on a given surface or curve. The same distinction is to be drawn in the case of the *analysis situs* to be discussed presently.

The investigations in both subjects up to this time have been occupied mainly with transformations of the second kind. Since in these investigations the question has not been with regard to the geometry on the surface or curve, but rather to find the criteria for the transformability of two surfaces or curves into each other, they are to be excluded from the sphere of the investigations here to be considered.† For

* [Groups with a finite number of parameters having been treated in the examples hitherto taken up, the so-called infinite groups will now be the subject of consideration in the text.]

† [From another point of view they are brought back again, which I did not yet know in 1872, very nicely into connection with the considerations in the text. Given any algebraic configuration (curve, or surface, etc.), let it be transferred into a higher space by introducing the ratios

$$\phi_1 : \phi_2 : \ldots : \phi_p$$

of the intergrands of the first species belonging to it as homogeneous co-ordinates. In this space we have then simply to take the group of homogeneous linear transformations as a basis for our further considerations. See various articles by *Brill, Nöther,* and *Weber,* and (to mention a single recent article) my own paper : *Zur Theorie der Abelschen Functionen* in vol. 36 of the Math. Annalen.]

the general synopsis here outlined does not embrace the entire field of mathematical research, but only brings certain lines of thought under a common point of view.

Of such a geometry of rational transformations as must result on the basis of the transformations of the first kind, only a beginning has so far been made. In the region of the first grade, viz., on the straight line, the rational transformations are identical with the linear transformations and therefore furnish nothing new. In the plane we know the totality of rational transformations (the Cremona transformations); we know that they can be produced by a combination of quadratic transformations. We know further certain invariant properties of plane curves [with reference to the totality of rational transformations], viz., their deficiency, the existence of moduli; but these considerations have not yet been developed into a geometry of the plane, properly speaking, in the meaning here intended. In space the whole theory is still in its infancy. We know at present but few of the rational transformations, and use them to establish correspondences between known and unknown surfaces.

2. *Analysis situs.*

In the so-called analysis situs we try to find what remains unchanged under transformations resulting from a combination of infinitesimal distortions. Here, again, we must distinguish whether the whole region, all space, for instance, is to be subjected to the transformations, or only a manifoldness contained in the same, a surface. It is the transformations of the first kind on which we could found a space geometry. Their group would be entirely different in constitution from the groups heretofore considered. Embracing as it does all transformations compounded from (real) infinitesimal point-transformations, it necessarily involves the limitation to real space-elements, and belongs to the domain of arbitrary functions. This group of transformations can be extended to advantage by combining it with those real collineations which at the same time affect the region at infinity.

3. *The Group of all Point-transformations.*

While with reference to this group no surface possesses any individual characteristics, as any surface can be converted into any other by transformations of the group, the group can be employed to advantage in the investigation of higher configurations. Under the view of geometry upon which we have taken our stand, it is a matter of no importance that these configurations have hitherto not been regarded as geo-

metric, but only as analytic, configurations, admitting occasionally of geometric application, and, furthermore, that in their investigation methods have been employed (these very point-transformations, for instance) which we have only recently begun to consciously regard as geometric transformations. To these analytic configurations belong, above all, homogeneous differential expressions, and also partial differential equations. For the general discussion of the latter, however, as will be explained in detail in the next section, the more comprehensive group of all contact-transformations seems to be more advantageous.

The principal theorem in force in the geometry founded on the group of all point-transformations is this: *that for an infinitesimal portion of space a point-transformation always has the value of a linear transformation.* Thus the developments of projective geometry will have their meaning for infinitesimals; and, whatever be the choice of the group for the treatment of the manifoldness, *in this fact lies a distinguishing characteristic of the projective view.*

Not having spoken for some time of the relation of methods of treatment founded on groups, one of which includes the other, let us now give one more example of the general theory of § 2. We will consider the question how projective properties are to be understood from the point of view of "all point-transformations," disregarding here the dualistic transformations which, properly speaking, form part of the group of projective geometry. This question is identical with the other question, What condition differentiates the group of linear point-transformations from the totality of point-transformations? What characterizes the linear group is this, that to every plane it makes correspond a plane; it contains those transformations under which the manifoldness of planes (or, what amounts to the same thing, of straight lines) remains unchanged. *Projective geometry is to be obtained from the geometry of all point-transformations by adjoining the manifoldness of planes, just as elementary is obtained from projective geometry by adjoining the imaginary circle at infinity.* Thus, for instance, from the point of view of all point-transformations the designation of a surface as an algebraic surface of a certain order must be regarded as an invariant relation to the manifoldness of planes. This becomes very clear if we connect, as *Grassmann* (Crelle's Journal, vol. 44) does, the generation of algebraic configurations with their construction by lines.

§ 9.

On the Group of all Contact-transformations.

Particular cases of contact-transformations have been long known; *Jacobi* has even made use of the most general contact-transformations in analytical investigations, but an effective geometrical interpretation has only been given them by recent researches of *Lie*'s.* It will therefore not be superfluous to explain here in detail what a contact-transformation is. In this we restrict ourselves, as hitherto, to point-space with its three dimensions.

By a contact-transformation is to be understood, analytically speaking, any substitution which expresses the values of the variables x, y, z and their partial derivatives $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$ in terms of new variables x', y', z', p', q'. It is evident that such substitutions, in general, convert surfaces that are in contact into surfaces in contact, and this accounts for the name. Contact-transformations are divided into three classes (the point being taken as space-element), viz., those in which *points* correspond to the ∞^3 points (the point-transformations just considered); those converting the points into curves; lastly, those converting them into surfaces. This classification is not to be regarded as essential, inasmuch as for other ∞^3 space-elements, say for planes, while a division into three classes again occurs, it does not coincide with the division occurring under the assumption of points as elements.

If a point be subjected to all the contact-transformations it is converted into the totality of points, curves, and surfaces. Only in their entirety, then, do points, curves, and surfaces form a *body* of our group. From this may be deduced the general rule that the formal treatment of a problem from the point of view of all contact-transformations (e.g., the theory of partial differential equations considered below) must be incomplete if we operate only with point- (or plane-) co-ordinates, for the very reason that the chosen space-elements do not form a body.

If, however, we wish to preserve the connection with the ordinary methods, it will not do to introduce as space-elements all the individual configurations contained in the body, as their number is ∞^∞. This makes it necessary to

* See, in particular, the article already cited: *Ueber partielle Differentialgleichungen und Complexe*, Mathematische Annalen, vol. 5. For the details given in the text in regard to partial differential equations I am indebted mainly to oral communications of *Lie*'s; see his note, *Zur Theorie partieller Differentialgleichungen*, Göttinger Nachrichten, October 1872.

introduce in these considerations as space-element not the point, curve, or surface, but the "surface-element," i.e., the system of values x, y, z, p, q. Each contact-transformation converts every surface-element into another; the ∞^5 surface-elements accordingly form a body.

From this point of view point, curve, and surface must be uniformly regarded as aggregates of surface-elements, and indeed of ∞^2 elements. For the surface is covered by ∞^2 elements, the curve is tangent to the same number, through the point pass the same number. But these aggregates of ∞^2 elements have another characteristic property in common. Let us designate as the *united position* of two consecutive surface-elements x, y, z, p, q and $x + dx$, $y + dy$, $z + dz$, $p + dp$, $q + dq$ the relation defined by the equation

$$dz - pdx - qdy = 0.$$

Thus point, curve, and surface agree in being *manifoldnesses of ∞^2 elements, each of which is united in position with the ∞^1 adjoining elements.* This is the common characteristic of point, curve, and surface; and this must serve as the basis of the analytical investigation, if the group of contact-transformations is to be used.

The united position of consecutive elements is an invariant relation under any contact-transformation whatever. And, conversely, contact-transformations may be defined as *those substitutions of the five variables x, y, z, p, q, by which the relation*

$$dz - pdx - qdy = 0$$

is converted into itself. In these investigations space is therefore to be regarded as a manifoldness of five dimensions; and this manifoldness is to be treated by taking as fundamental group the totality of the transformations of the variables which leave a certain relation between the differentials unaltered.

First of all present themselves as subjects of investigation the manifoldnesses defined by one or more equations between the variables, i.e., *by partial differential equations of the first order, and systems of such equations.* It will be one of the principal problems to select out of the manifoldnesses of elements satisfying given equations systems of ∞^1, or of ∞^2, elements which are all united in position with a neighboring element. A question of this kind forms the sum and substance of the problem of the solution of a partial differential equation of the first order. It can be formulated in the following way: to select from among the ∞^4 elements satisfying the equation all the twofold manifoldnesses of the given

kind. The problem of the complete solution thus assumes the definite form: to classify in some way the ∞^4 elements satisfying the equation into ∞^3 manifoldnesses of the given kind.

It cannot be my intention to pursue this consideration of partial differential equations further; on this point I refer to *Lie*'s articles already cited. I will only point out one thing further, that from the point of view of the contact-transformations a partial differential equation of the first order has no invariant, that every such equation can be converted into any other, and that therefore linear equations in particular have no distinctive properties. Distinctions appear only when we return to the point of view of the point-transformations.

The groups of contact-transformations, of point-transformations, finally of projective transformations, may be defined in a uniform manner which should here not be passed over.* Contact-transformations have already been defined as those transformations under which the united position of consecutive surface-elements is preserved. But, on the other hand, point-transformations have the characteristic property of converting consecutive line-elements which are united in position into line-elements similarly situated; and, finally, linear and dualistic transformations maintain the united position of consecutive connex-elements. By a connex-element is meant the combination of a surface-element with a line-element contained in it; consecutive connex-elements are said to be united in position when not only the point but also the line-element of one is contained in the surface-element of the other. The term connex-element (though only preliminary) has reference to the configurations recently introduced into geometry by *Clebsch* † and represented by an equation containing simultaneously a series of point-coordinates as well as a series of plane- and a series of line-coordinates whose analogues in the plane *Clebsch* denotes as connexes.

§ 10.

On Manifoldnessess of any Number of Dimensions.

We have already repeatedly laid stress on the fact that in connecting the expositions thus far with space-perception we have only been influenced by the desire to be able to develop abstract ideas more easily through dependence on graphic

* I am indebted to a remark of *Lie*'s for these definitions.

† Göttinger Abhandlungen, 1872 (vol. 17): *Ueber eine Fundamentalaufgabe der Invariantentheorie*, and especially Göttinger Nachrichten, 1872, No. 22: *Ueber ein neues Grundgebilde der analytischen Geometrie der Ebene.*

examples. But the considerations are in their nature independent of the concrete image, and belong to that general field of mathematical research which is designated as the theory of manifoldnesses of any dimensions,—called by *Grassmann* briefly "theory of extension" (Ausdehnungslehre). How the transference of the preceding development from space to the simple idea of a manifoldness is to be accomplished is obvious. It may be mentioned once more in this connection that in the abstract investigation we have the advantage over geometry of being able to choose arbitrarily the fundamental group of transformations, while in geometry a minimum group—the principal group—was given at the outset.

We will here touch, and that very briefly, only on the following three methods:

1. *The Projective Method or Modern Algebra* (*Theory of Invariants*).

Its group consists of the totality of linear and dualistic transformations of the variables employed to represent individual configurations in the manifoldness; it is the generalization of projective geometry. We have already noticed the application of this method in the discussion of infinitesimals in a manifoldness of one more dimension. It includes the two other methods to be mentioned, in so far as its group includes the groups upon which those methods are based.

2. *The Manifoldness of Constant Curvature.*

The notion of such a manifoldness arose in *Riemann's* theory from the more general idea of a manifoldness in which a differential expression in the variables is given. In his theory the group consists of the totality of those transformations of the variables which leave the given expression unchanged. On the other hand, the idea of a manifoldness of constant curvature presents itself when a projective measurement is based upon a given quadratic equation between the variables. From this point of view as compared with *Riemann's* the extension arises that the variables are regarded as complex; the variability can be limited to the real domain afterwards. Under this head belong the long series of investigations touched on in §§ 5, 6, 7.

3. *The Plane Manifoldness.*

Riemann designates as a plane manifoldness one of constant zero curvature. Its theory is the immediate generalization of elementary geometry. Its group can, like the princi-

pal group of geometry, be separated from out the group of the projective method by supposing a configuration to remain fixed which is defined by two equations, a linear and a quadratic equation. We have then to distinguish between real and imaginary if we wish to adhere to the form in which the theory is usually presented. Under this head are to be counted, in the first place, elementary geometry itself, then for instance the recent generalizations of the ordinary theory of curvature, etc.

Concluding Remarks.

In conclusion we will introduce two further remarks closely related to what has thus far been presented,—one with reference to the analytic form in which the ideas developed in the preceding pages are to be represented, the other marking certain problems whose investigation would appear important and fruitful in the light of the expositions here given.

Analytic geometry has often been reproached with giving preference to arbitrary elements by the introduction of the system of co-ordinates, and this objection applies equally well to every method of treating manifoldnesses in which individual configurations are characterized by the values of variables. But while this objection has been too often justified owing to the defective way in which, particularly in former times, the method of co-ordinates was manipulated, yet it disappears when the method is rationally treated. The analytical expressions arising in the investigation of a manifoldness with reference to its group must, from their meaning, be independent of the choice of the co-ordinate system; and the problem is then to clearly set forth this independence analytically. That this can be done, and how it is to be done, is shown by modern algebra, in which the abstract idea of an invariant that we have here in view has reached its clearest expression. It possesses a general and exhaustive law for constructing invariant expressions, and operates only with such expressions. This object should be kept in view in any formal (analytical) treatment, even when other groups than the projective group form the basis of the treatment.* For the analytical formulation should, after all, be congruent with the conceptions whether it be our purpose to use it only as a precise and perspicuous expression of the conceptions, or to penetrate by its aid into still unexplored regions.

The further problems which we wished to mention arise on comparing the views here set forth with the so-called *Galois* theory of equations.

* [For instance, in the case of the groups of rotations of three-dimensional space about a fixed point, such a formalism is furnished by quaternions.]

In the *Galois* theory, as in ours, the interest centres on groups of transformations. The objects to which the transformations are applied are indeed different; there we have to do with a finite number of discrete elements, here with the infinite number of elements in a continuous manifoldness. But still the comparison may be pursued further owing to the identity of the group-idea,* and I am the more ready to point it out here, as it will enable us to characterize the position to be awarded to certain investigations begun by *Lie* and myself † in accordance with the views here developed.

In the Galois theory, as it is presented for instance in *Serret's* "Cours d'Algèbre supérieure" or in *C. Jordan's* "Traité des Substitutions," the real subject of investigation is the group theory or substitution theory itself, from which the theory of equations results as an application. Similarly we require a *theory of transformations*, a theory of the groups producible by transformations of any given characteristics. The ideas of commutativity, of similarity, etc., will find application just as in the theory of substitutions. As an application of the theory of transformations appears that treatment of a manifoldness which results from taking as a basis the groups of transformations.

In the theory of equations the first subjects to engage the attention are the symmetric functions of the coefficients, and in the next place those expressions which remain unaltered, if not under all, yet under a considerable number of permutations of the roots. In treating a manifoldness on the basis of a group our first inquiry is similarly with regard to the bodies (§ 5), viz., the configurations which remain unaltered under all the transformations of the group. But there are configurations admitting not all but some of the transformations of the group, and they are next of particular interest from the point of view of the treatment based on the group; they have distinctive characteristics. It amounts, then, to distinguishing in the sense of ordinary geometry symmetric and regular bodies, surfaces of revolution and helicoidal surfaces. If the subject be regarded from the point of view of projective geometry, and if it be further required that the transformations converting the configurations into themselves shall be commutative, we arrive at the configurations considered by *Lie* and myself in the article cited, and the general problem proposed in § 6 of that article. The deter-

* I should like here to call to mind *Grassmann's* comparison of combinatory analysis and extensive algebra in the introduction to the first edition of his "Ausdehnungslehre" (1844).

† See our article: *Ueber diejenigen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen*, Mathematische Annalen, Bd. IV.

mination (given in §§ 1, 3 of that article) of all the groups of an infinite number of commutative linear transformations in the plane forms a part of the general theory of transformations named above.*

NOTES.

I. *On the Antithesis between the Synthetic and the Analytic Method in Modern Geometry.*

The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We choose therefore in the text as common designation of them both the term *projective geometry*. Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulæ of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. On the other hand, the advantage to original research of a well formulated analysis should not be underestimated,—an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important, step.

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* I must refrain from referring in the text to the fruitfulness of the consideration of infinitesimal transformations in the theory of differential equations. In § 7 of the article cited, *Lie* and I have shown that ordinary differential equations which admit the same infinitesimal transformations present like difficulties of integration. How the considerations are to be employed for partial differential equations, *Lie* has illustrated by various examples in several places; for instance, in the article named above (Math. Annalen, vol. 5). See in particular the proceedings of the Christiania Academy, May 1872.

[At this time I may be allowed to refer to the fact that it is exactly the two problems mentioned in the text which have influenced a large part of the further investigations of *Lie* and myself. I have already called attention to the appearance of the two first volumes of *Lie*'s "Theorie der Transformationsgruppen." Of my own work might be mentioned the later researches on regular bodies, on elliptic modular functions, and on single-valued functions with linear transformations into themselves, in general. An account of the first of these was given in a special work: "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen fünften Grades" (Leipzig, 1884); an exposition of the theory of the elliptic modular functions, elaborated by *Dr. Fricke* is in course of publication.]

II. *Division of Modern Geometry into Theories.*

When we consider, for instance, how persistently the mathematical physicist disregards the advantages afforded him in many cases by only a moderate cultivation of the projective view, and how, on the other hand, the student of projective geometry leaves untouched the rich mine of mathematical truths brought to light by the theory of the curvature of surfaces, we must regard the present state of mathematical knowledge as exceedingly incomplete and, it is to be hoped, as transitory.

III. *On the Value of Space-perception.*

When in the text we designated space-perception as something incidental, we meant this with regard to the purely mathematical contents of the ideas to be formulated. Space-perception has then only the value of illustration, which is to be estimated very highly from the pedagogical stand-point, it is true. A geometric model, for instance, is from this point of view very instructive and interesting.

But the question of the value of space-perception in itself is quite another matter. I regard it as an independent question. There is a true geometry which is not, like the investigations discussed in the text, intended to be merely an illustrative form of more abstract investigations. Its problem is to grasp the full reality of the figures of space, and to interpret—and this is the mathematical side of the question—the relations holding for them as evident results of the axioms of space-perception. A model, whether constructed and observed or only vividly imagined, is for this geometry not a means to an end, but the subject itself.

This presentation of geometry as an independent subject, apart from and independent of pure mathematics, is nothing new, of course. But it is desirable to lay stress explicitly upon this point of view once more, as modern research passes it over almost entirely. This is connected with the fact that, *vice versa*, modern research has seldom been employed in investigations on the form-relations of space-configurations, while it appears well adapted to this purpose.

IV. *On Manifoldnesses of any Number of Dimensions.*

That space, regarded as the locus of points, has only three dimensions, does not need to be discussed from the mathematical point of view; but just as little can anybody be prevented from that point of view from claiming that space really has four, or an unlimited number of dimensions, and that we are only able to perceive three. The theory of mani-

foldnesses, advancing as it does with the course of time more and more into the foreground of modern mathematical research, is by its nature fully independent of any such claim. But a nomenclature has become established in this theory which has indeed been derived from this idea. Instead of the elements of a manifoldness we speak of the points of a higher space, etc. The nomenclature itself has certain advantages, in that it facilitates the interpretation by calling to mind the perceptions of geometry. But it has had the unfortunate result of causing the wide-spread opinion that investigations on manifoldnesses of any number of dimensions are inseparably connected with the above-mentioned idea of the nature of space. Nothing is more unsound than this opinion. The mathematical investigations in question would, it is true, find an immediate application to geometry, if the idea were correct; but their value and purport is absolutely independent of this idea, and depends only on their own mathematical contents.

It is quite another matter when *Plücker* shows how to regard actual space as a manifoldness of any number of dimensions by introducing as space-element a configuration depending on any number of parameters, a curve, surface, etc. (see § 5 of the text).

The conception in which the element of a manifoldness (of any number of dimensions) is regarded as analogous to the point in space was first developed, I suppose, by *Grassmann* in his "Ausdehnungslehre" (1844). With him the thought is absolutely free of the above-mentioned idea of the nature of space; this idea goes back to occasional remarks by *Gauss*, and became more widely known through *Riemann*'s investigations on manifoldnesses, with which it was interwoven.

Both conceptions—*Grassmann*'s as well as *Plücker*'s—have their own peculiar advantages; they can be alternately employed with good results.

V. *On the So-called Non-Euclidean Geometry.*

The projective metrical geometry alluded to in the text is essentially coincident, as recent investigations have shown, with the metrical geometry which can be developed under non-acceptance of the axiom of parallels, and is to-day under the name of non-Euclidean geometry widely treated and discussed. The reason why this name has not been mentioned at all in the text, is closely related to the expositions given in the preceding note. With the name non-Euclidean geometry have been associated a multitude of non-mathematical ideas, which have been as zealously cherished by some as resolutely rejected by others, but with which our purely mathematical

considerations have nothing to do whatever. A wish to contribute towards clearer ideas in this matter has occasioned the following explanations.

The investigations referred to on the theory of parallels, with the results growing out of them, have a definite value for mathematics from two points of view.

They show, in the first place,—and this function of theirs may be regarded as concluded once for all,—that the axiom of parallels is not a mathematical consequence of the other axioms usually assumed, but the expression of an essentially new principle of space-perception, which has not been touched upon in the foregoing investigations. Similar investigations could and should be performed with regard to every axiom (and not alone in geometry); an insight would thus be obtained into the mutual relation of the axioms.

But, in the second place, these investigations have given us an important mathematical idea,—the idea of a manifoldness of constant curvature. This idea is very intimately connected, as has already been remarked and in § 10 of the text discussed more in detail, with the projective measurement which has arisen independently of any theory of parallels. Not only is the study of this measurement in itself of great mathematical interest, admitting of numerous applications, but it has the additional feature of including the measurement given in geometry as a special (limiting) case and of teaching us how to regard the latter from a broader point of view.

Quite independent of the views set forth is the question, what reasons support the axiom on parallels, i.e., whether we should regard it as absolutely given, as some claim, or only as approximately proved by experience, as others say. Should there be reasons for assuming the latter position, the mathematical investigations referred to afford us then immediately the means for constructing a more exact geometry. But the inquiry is evidently a philosophical one and concerns the most general foundations of our understanding. The mathematician as such is not concerned with this inquiry, and does not wish his investigations to be regarded as dependent on the answer given to the question from the one or the other point of view.*

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* [To the explanations in the text I should like to add here two supplementary remarks.

In the first place, when I say that the mathematician as such has no stand to take on the philosophical question, I do not mean to say that the philosopher can dispense with the mathematical developments in treating the aspect of the question which interests him; on the contrary, it is my decided conviction that a study of these developments is

VI. *Line Geometry as the Investigation of a Manifoldness of Constant Curvature.*

In combining line geometry with the projective measurement in a manifoldness of five dimensions, we must remember that the straight lines represent elements of the manifoldness which, metrically speaking, are at infinity. It then becomes necessary to consider what the value of a system of projective measurement is for the elements at infinity; and this may here be set forth somewhat at length, in order to remove any difficulties which might else seem to stand in the way of conceiving of line geometry as a metrical geometry. We shall illustrate these expositions by the graphic example of the projective measurement based on a quadric surface.

Any two points in space have with respect to the surface an absolute invariant,—the anharmonic ratio of the two points together with the two points of intersection of the line joining them with the surface. But when the two points move up to the surface, this anharmonic ratio becomes zero independently of the position of the points, except in the case where the two points fall upon a generator, when it becomes indeterminate. This is the only special case which can occur in their relative position unless they coincide, and we have therefore the theorem:

The projective measurement in space based upon a quadric surface does not yet furnish a measurement for the geometry on the surface.

This is connected with the fact that by linear transformations of the surface into itself any three points of the surface can be brought into coincidence with three others.*

If a measurement on the surface itself be desired, we must limit the group of transformations, and this result is obtained by supposing any arbitrary point of space (or its polar plane) to be fixed. Let us first take a point not on the surface. We can then project the surface from the point upon a plane, when a conic will appear as the boundary curve. Upon this

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the indispensable prerequisite to every philosophical discussion of the subject.

Secondly, I have not meant to say that my *personal* interest is exhausted by the mathematical aspect of the question. For my conception of the subject, in general, let me refer to a recent paper: "Zur Nicht-Euklidischen Geometrie" (Math. Annalen, vol. 37).]

* These relations are different in ordinary metrical geometry; for there it is true that two points at infinity have an absolute invariant. The contradiction which might thus be found in the enumeration of the linear transformations of the surface at infinity into itself is removed by the fact that the translations and transformations of similarity contained in this group do not alter the region at infinity at all.

conic we can base a projective measurement in the plane, which must then be transferred back to the surface.* This is a measurement with constant curvature in the true sense, and we have then the theorem :

Such a measurement on the surface is obtained by keeping fixed a point not on the surface.

Correspondingly, we find : †

A measurement with zero curvature on the surface is obtained by choosing as the fixed point a point of the surface itself.

In all these measurements on the surface the generators of the surface are lines of zero length. The expression for the element of arc on the surface differs therefore only by a factor in the different cases. There is no absolute element of arc upon the surface; but we can of course speak of the angle formed by two directions on the surface.

All these theorems and considerations can now be applied immediately to line geometry. Line-space itself admits at the outset no measurement, properly speaking. A measurement is only obtained by regarding a linear complex as fixed; and the measurement is of constant or zero curvature, according as the complex is a general or a special one (a line). The selection of a particular complex carries with it further the acceptation of an absolute element of arc. Independently of this, the directions to adjoining lines cutting the given line are of zero length, and we can besides speak of the angle between any two directions.‡

VII. *On the Interpretation of Binary Forms.*

We shall now consider the graphic illustration which can be given to the theory of invariants of binary cubics and biquadratics by taking advantage of the representation of $x + iy$ on the sphere.

A binary cubic f has a cubic covariant Q, a quadratic covariant Δ, and an invariant R.§ From f and Q a whole system of covariant sextics $Q^2 + \lambda R f^2$ may be compounded, among them being Δ^2. It can be shown ‖ that every covariant of the cubic must resolve itself into such groups of six points.

* See § 7 of the text.

† See § 4 of the text.

‡ See the article *Ueber Liniengeometrie und metrische Geometrie*, Math. Annalen, vol. 5, p. 271.

§ See in this connection the corresponding sections of Clebsch's "Theorie der binären Formen."

‖ By considering the linear transformations of f into itself. See Math. Annalen, vol. 4, p. 352.

Inasmuch as λ can assume complex values, the number of these covariants is ∞^2.*

The whole system of forms thus defined can now be represented upon the sphere as follows. By a suitable linear transformation of the sphere into itself let the three points representing f be converted into three equidistant points of a great circle. Let this great circle be denoted as the equator, and let the three points f have the longitudes 0°, 120°, 240°. Then Q will be represented by the points of the equator whose longitudes are 60°, 180°, 300°; Δ by the two poles. Every form $Q^2 + \lambda R f^2$ is represented by six points, whose latitude and longitude are given in the following table, where α and β are arbitrary numbers:

α	α	α	$-\alpha$	$-\alpha$	$-\alpha$
β	$120° + \beta$	$240° + \beta$	$-\beta$	$120° - \beta$	$240° - \beta$

In studying the variation of these systems of points on the sphere, it is interesting to see how they give rise to f and Q (each reckoned twice) and Δ (reckoned three times).

A biquadratic f has a biquadratic covariant H, a sextic covariant T, and two invariants i and j. Particularly noteworthy is the pencil of biquadratic forms $iH + \lambda jf$, all belonging to the same T, among them being the three quadratic factors into which T can be resolved, each reckoned twice.

Let the centre of the sphere now be taken as the origin of a set of rectangular axes OX, OY, OZ. Their six points of intersection with the sphere make up the form T. The four points of a set $iH + \lambda jf$ are given by the following table, x, y, z being the co-ordinates of any point of the sphere :

$$\begin{array}{rrr} x, & y, & z, \\ x, & -y, & -z, \\ -x, & y, & -z, \\ -x, & -y, & z. \end{array}$$

The four points are in each case the vertices of a symmetrical tetrahedron, whose opposite edges are bisected by the co-ordinate axes ; and this indicates the rôle played by T in the theory of biquadratic equations as the resolvent of $iH + \lambda jf$.

ERLANGEN, *October*, 1872.

* [See Beltrami, *Ricerche sulla geometria delle forme binarie cubiche*, Memorie dell' Accademia di Bologna, 1870.]

THE TRANSITIVE SUBSTITUTION-GROUPS OF NINE LETTERS.

BY DR. F. N. COLE.

The list of these groups published by Mr. Askwith in the present volume (26) of the Quarterly Journal of Mathematics contains only 22 of the 34 actually existing types. The groups identified by Mr. Askwith are the more obvious forms—the symmetric and alternating groups and the non-primitives—together with two primitive types of order 72, while the omitted cases—5 non-primitives and 7 primitives—include among others such specially interesting forms as the triad group of order 216 identified with the theory of the points of inflection of the plane cubic curve, etc., and a *simple* group, apparently heretofore unrecognized, of order 504.*

I give here a complete list of the transitive groups of this degree, together with very brief explanations of the processes by which I have obtained them.

A. *The Non-primitive Groups of Nine Letters.*

In respect to these groups the nine letters are distributed in three systems of three letters each, such that every substitution of the group replaces every system either by itself or by one of the other systems.

Those substitutions of the group which replace every system by itself form a self-conjugate subgroup, i.e., a subgroup which is transformed into itself by every substitution of the group. This subgroup obviously affects the three systems symmetrically.

Of the intransitive groups of nine letters with three transitive systems of three letters each only the following 8 satisfy the last requirement :

$$H_{216} = (abc) \text{ all } (def) \text{ all } (ghi) \text{ all},$$
$$H_{108} = \{(abc) \text{ all } (def) \text{ all } (ghi) \text{ all}\} \text{ pos.},$$
$$H_{54} = (abc) \text{ pos. } (def) \text{ pos. } (ghi) \text{ pos.} + (abc) \text{ neg. } (def) \text{ neg. } (ghi) \text{ neg.},$$
$$H_{27} = (abc) \text{ pos. } (def) \text{ pos. } (ghi) \text{ pos.},$$

$H_{18} =$	1*,	1,	$abc\,.\,def$,	$acb\,.\,dfe$,
	ghi,	abc,	$acb\,.\,def$,	dfe,
	gih,	acb,	$abc\,.\,dfe$,	def,
	gh,	$ab\,.\,de$,	$ac\,.\,df$,	$bc\,.\,ef$,
	gi,	$ac\,.\,de$,	$bc\,.\,df$,	$ab\,.\,ef$,
	hi,	$bc\,.\,de$,	$ab\,.\,df$,	$ac\,.\,ef$,

* I use this notation to indicate that every substitution of e, f, g is multiplied by the 3 several substitutions opposite it.

$$H_9 = \begin{array}{l|lll} 1, & 1, & abc\,.\,def, & acb\,.\,dfe, \\ ghi, & abc, & acb\,.\,def, & dfe, \\ gih, & acb, & abc\,.\,dfe, & def, \end{array}$$

$$H_6 = 1,\quad \begin{array}{ll} abc\,.\,def\,.\,ghi, & ab\,.\,de\,.\,gh, \\ acb\,.\,dfe\,.\,gih, & ac\,.\,df\,.\,gi, \\ & bc\,.\,ef\,.\,hi, \end{array}$$

$$H_3 = (abc\,.\,def\,.\,ghi)\ \text{cyc.}$$

The non-primitive groups are formed by adding to the preceding groups H substitutions which permute the three systems a, b, c; d, e, f; g, h, i. These substitutions either permute the three systems cyclically, or they leave one system unchanged and interchange the other two. In order that the nine letters may be transitively connected, the cyclical permutations of the three systems must be present, and if there are any other permutations of the systems, all the six possible ones occur.

If H_r is the subgroup which in any particular case leaves all the systems unchanged, and if any substitution σ of the group permutes the systems in any way, then all the substitutions σH_r permute the systems in the same way, and these are all the substitutions of the group which produce this effect. Accordingly, the order of a non-primitive group of nine letters is either three or six times the order of the corresponding H_r.

Given the group H_r, a substitution of the required non-primitive group which shall permute the systems of non-primitivity cyclically may be selected in any way consistent with this requirement, subject only to the two conditions 1) that it must transform H_r into itself and 2) that its third power (which leaves the systems unchanged) must occur in H_r. To the resulting transitive group G of order $3r$ a further substitution may then be added which interchanges two of the three systems, which transform G into itself, and whose second power occurs in H_r.

I find the following 23 distinct types of non-primitive groups with nine letters:

Order

1296 $\{H_{216}, adg\,.\,beh\,.\,cfi, ad\,.\,be\,.\,cf\}$.

648_1 $\{H_{216}, adg\,.\,beh\,.\,cfi\}$.

$_2$ $\{H_{108}, adg\,.\,beh\,.\,cfi, adbe\,.\,cf\}$.

$_3$ $\{H_{108}, adg\,.\,beh\,.\,cfi, ad\,.\,be\,.\,cf\}$.

324_1 $\{H_{108}, adg\,.\,beh\,.\,cfi\}$.

$_2$ $\{H_{54}, adg\,.\,beh\,.\,cfi, ad\,.\,bf\,.\,ce\,.\,gh\}$.

162_1 $\{H_{54},\ adg\,.\,beh\,.\,cfi\}$.

$_2$ $\{H_{27},\ adg\,.\,beh\,.\,cfi,\ ad\,.\,bf\,.\,ce\,.\,gh\}$.

$_3$ $\{H_{27},\ adg\,.\,beh\,.\,cfi,\ ad\,.\,be\,.\,cf\}$.

108 $\{H_{18},\ adg\,.\,bfi\,.\,ceh,\ ad\,.\,be\,.\,cf\,.\,gh\}$.

81 $\{H_{27},\ adg\,.\,bfi\,.\,ceh\}$.

54_1 $\{H_{18},\ adg\,.\,bfi\,.\,ceh\}$.

$_2$ $\{H_9,\ adgbficeh,\ ad\,.\,be\,.\,cf\,.\,gh\}$.

$_3$ $\{H_9,\ adg\,.\,bfi\,.\,ceh,\ ad\,.\,be\,.\,cf\,.\,gh\}$.

$_4$ $\{H_9,\ adg\,.\,bfi\,.\,ceh,\ ad\,.\,bf\,.\,ce\}$.

36 $\{H_6,\ adg\,.\,beh\,.\,cfi,\ ad.\,bf\,.\,ce\,.\,hi\}$.

27_1 $\{H_9,\ adgbficeh\}$.

$_2$ $\{H_9,\ adg\,.\,bfi\,.\,ceh\}$.

18_1 $\{H_6,\ adg\,.\,beh\,.\,cfi\}$.

$_2$ $\{(abcdefghi)$ cyc., $bi\,.\,ch\,.\,dg\,.\,ef\}$.

$$_3\ \left\{\begin{array}{l} abc\,.\,def\,.\,ghi, \\ adg\,.\,beh\,.\,cfi, \\ aec\,.\,bfg\,.\,cdh, \\ afh\,.\,bde\,.\,ceg, \end{array}\ ad\,.\,bf\,.\,ce\,.\,hi\right\} = \{(abcdefghi)_9,\ ad\,.\,bf\,.\,ce\,.\,hi\}.$$

9_1 $(abcdefghi)$ cyc.

$_2$ $(abcdefghi)_9$.

B. *The Primitive Groups of Nine Letters.*

In a primitive group the subgroup which leaves any letter unchanged affects all the remaining letters.* These subgroups may therefore in the present case be selected from the table of groups of eight letters. The number of possibilities is greatly reduced by the consideration that a primitive group cannot contain a substitution of two or of three letters, or of four letters, if it affects more than eight letters,† and that, if it contains a transitive subgroup of lower degree, it is at least doubly transitive.‡

With these restrictions the only groups of eight letters, except the alternating and symmetric groups, which can serve as bases for primitive groups of nine letters are readily found to be the following:

Order 336, 168_1, 168_2, 56, 48, 24, 24, 16, 16, 8_{20}, 8_{21}, 8_{22}, 8_{23}, 8_{24}, 8_{25}, 6_2, 6_3, 6_4, 4_6, 4_8, 2.

The orders of the corresponding primitive groups will be these multiplied by nine.

* Cf. Netto, Theory of Substitutions, American Edition, p. 94.
† Ibid., p. 138. ‡ Ibid., p. 95.

The several possibilities have now to be considered separately.

Order 9 . 336. This group would contain 36 conjugate subgroups of order 7, each of which would therefore be transformed into itself by a group of $9 \cdot 336 \div 36 = 84$ substitutions. The latter group is transitive in 7 letters, but not in 8 or in 9. It could only be formed as $\{(abcdefg)_{42}, (hi)\}$. But then the required group would contain the transposition (hi). *There is therefore no transitive group of this order in nine letters.*

Order 9 . 168. This group would contain 36 conjugate subgroups of order 7, each transformed into itself by a group of 42 substitutions. The latter is transitive in 7 letters, but not in 8 or in 9. It could only be found as a transitive H_{42} of 7 letters dimidiated with the H_2 of the 2 remaining letters. The required group will therefore contain substitutions with one cycle of 6 letters and one of 2 letters. *This can happen only with the compound H_{168} of 8 letters.**

The latter contains the substitution of order 7

$$\sigma = bcedghf.$$

The transitive subgroup of order 42 with 7 letters, of which this is a part, contains the substitution

$$cf \,.\, eh \,.\, dg.$$

Accordingly the required group of order 9 . 168 contains the substitution

$$s = ai \,.\, cf \,.\, eh \,.\, dg.$$

The H_{168} of 8 letters contains also

$$t = ac \,.\, bd \,.\, eg \,.\, fh,$$

and consequently the required group contains

$$st = aichgbdef = \rho.$$

The H_{168} of 8 letters contains

$$H_{56} = \{A(abcdefgh),\ bcedghf\}.$$

We proceed now to show that for every α

$$\rho^\alpha H_{56} = H_{56}\rho^\beta,$$

and that consequently ρ and H_{56} generate a (transitive) group of 9 letters of order 504. Letting

$$\begin{array}{lll} \tau_2 = ab \,.\, cd \,.\, ef \,.\, gh, & \tau_5 = ae \,.\, bf \,.\, cg \,.\, dh, & \tau_8 = ah \,.\, bg \,.\, cf \,.\, de, \\ \tau_3 = ac \,.\, bd \,.\, eg \,.\, fh, & \tau_6 = af \,.\, be \,.\, ch \,.\, dg, & \sigma = bcedghf, \\ \tau_4 = ad \,.\, bc \,.\, eh \,.\, fg, & \tau_7 = ag \,.\, bh \,.\, ce \,.\, df, & \rho = aichgbdef, \end{array}$$

* Cf. Hölder, Math. Annalen, vol. 40, p. 82.

I find

$$\begin{array}{lll}
\rho\,\sigma = \sigma\tau_7\rho^6, & \rho\,\tau_2 = \sigma\tau_3\rho^5, & \rho\,\tau_5 = \sigma^2\tau_8\rho^3, \\
\rho^2\sigma = \sigma^6\rho^7, & \rho^2\tau_2 = \sigma^4\tau_8\rho^3, & \rho^2\tau_5 = \sigma^2\tau_6\rho^5, \\
\rho^3\sigma = \sigma^6\tau_4\rho^3, & \rho^3\tau_2 = \sigma^3\tau_3\rho^2, & \rho^3\tau_5 = \sigma^5\tau_3\rho, \\
\rho^4\sigma = \sigma^5\tau_3\rho, & \rho^4\tau_2 = \sigma^6\tau_7\rho^6, & \rho^4\tau_5 = \sigma^4\tau_7\rho^7, \\
\rho^5\sigma = \tau_6\rho^3, & \rho^5\tau_2 = \sigma^6\tau_8\rho, & \rho^5\tau_5 = \sigma^4\tau_4\rho^2, \\
\rho^6\sigma = \sigma^5\tau_6\rho^5, & \rho^6\tau_2 = \sigma^3\tau_7\rho^7, & \rho^6\tau_5 = \sigma^5\tau_7\rho^8, \\
\rho^7\sigma = \sigma^2\tau_3\rho^4, & \rho^7\tau_2 = \sigma^4\tau_3\rho^6, & \rho^7\tau_5 = \sigma^2\tau_3\rho^4, \\
\rho^8\sigma = \sigma^6\tau_4\rho^8, & \rho^8\tau_2 = \sigma\tau_8\rho^4, & \rho^8\tau_5 = \sigma^2\tau_8\rho^6, \\
 & \rho^a\tau_3 = \tau_3\rho^{-a}. &
\end{array}$$

Since, now, τ_2, τ_3, and τ_5 generate the group $A(abcdefgh)_8$, and this in combination with σ generates the group H_{56}, the preceding relations are sufficient to insure that $\rho^a H_{56} = H_{56}\rho^\beta$.

There is therefore a group of order 504 in 9 letters. If the group of order 9 . 168 exists, it must contain the group of order 504, and beside this, for example, the substitution

$$\omega = ceg\,.\,dfh.$$

The latter transforms σ into σ^2. It transforms $A(abcdefgh)_8$ into itself. It transforms ρ into $aiedcbfgh = \rho\tau_2$. Consequently it transforms the subgroup of order 504 into itself. Then in combination with the latter it generates the required group of order 1512.

There is one and only one type of transitive group of order 9 . 168 *with* 9 *letters. This is obtained by adding to the compound* H_{168} *of* 8 *letters* $[=\{(bcedghf)_{21},\ A(abcdefgh)\}]$ *the substitution aichgbdef.*

Order 9 . 56. This has already been treated under the preceding case.

There is one and only one type of transitive group of order 504 *in* 9 *letters, viz., the combination of the* H_{56} *of* 8 *letters with, for example, aichgbdef.*

This group is particularly noteworthy in view of the fact that it is *simple*, i.e., that it contains no self-conjugate subgroup.* Its substitutions which affect 9 letters form 28 cyclical subgroups of order 9, no two of which have any substitution, except identity, in common.

Order 9 . 48. This group is based on the H_{48} of eight letters

$$\{ABC = ade\,.\,bcf,\quad ABCD = acfhbdeg\}.$$

Its even substitutions form a self-conjugate subgroup of order 9 . 24 based on the H_{24} of eight letters

$$\{ABC = ade\,.\,bcf,\quad AC\,.\,BD = afbe\,.\,chdg\}.$$

* Below order 660 the only simple groups of compound order are one group for each of the orders 60, 168, 360, and 504. Cf. Hölder: Math. Ann., vol. 40, p. 55, and Cole: Amer. Jour. of Math., vol. 14, p. 378.

The latter contains 1 or 4 conjugate subgroups of order 27.* In the former case the subgroup of order 27 is transitive,† and those of its substitutions which leave a single letter unchanged form a subgroup of order 3. This subgroup must be self-conjugate within the H_{24} of the 8 letters affected. This being impossible, it follows that there are 4 conjugate subgroups of order 27.

The substitutions of the group of order 9.24 transform these 4 subgroups in 24, 12, 8, or 4 ways. Accordingly the 4 subgroups are all transformed into themselves by 9, 18, 27, or 54 substitutions, which form a self-conjugate subgroup‡ of the group of order 9.24. The first two cases both imply a self-conjugate subgroup of order 9; the last two one of order 27. The required group of order 9.24 must therefore contain a self-conjugate subgroup of order 9.

This subgroup is transitive. It cannot be cyclical; for a brief consideration shows that it could not then be transformed into itself by $AC.BD = afbe.chdg$. Accordingly, it contains 4 subgroups of order 3. At least one of these $\{\sigma\}$ must then be transformed into itself by $ABC = ade.bcf$. We can take only

$$\sigma_1 = ade.bcf.\begin{cases} ghi \\ gih \end{cases}, \quad \sigma_2 = ade.bfc.\begin{cases} ghi \\ gih \end{cases}.$$

Of these σ_1 must be rejected since $\sigma_1^2.ade.bcf$ affects only 3 letters. Also $ACD = afg.beh$ transforms σ_2 into

$$\tau_2 = fdh.egc.\begin{cases} abi \\ bai \end{cases};$$

and

$$ade.bfc.ghi \times fdh.egc.abi = ah.bdgfe.$$

The last substitution obviously cannot occur in the required group. Consequently we must take

$$\sigma_2 = ade.bfc.gih, \quad \tau_2 = fdh.egc.bai.$$

These generate a group of order 9.

$$\begin{array}{llll} 1, & ade.bfc.gih, & aed.bcf.ghi, \\ & aib.ceg.fdh, & abi.cge.fhd, \\ & ahc.bdg.eif, & ach.bgd.efi, \\ & agf.beh.cdi, & afg.bhe.cid, \end{array}$$

and this group is transformed into itself by $ade.bcf$ and $afbe.chdg$ and therefore by the entire H_{24} of 8 letters. As

* Cf. Sylow: Math. Ann., vol. 5, p. 584–94.
† Cf. Netto: Theory of Substitutions, American edition, p. 82.
‡ Cf. Hölder, l. c., p. 57.

the H_9 and the H_{24} have no substitutions in common, their combination generates a group of order 216.

The latter is furthermore transformed into itself by $ABCD = acfhbdeg$, and therefore its combination with this substitution generates the required group of order 432.

There is a single type of primitive group of order 432 *of* 9 *letters. This is a doubly transitive group obtained by adding to the* $H_{48} = \{ABC = ade\,.\,bcf,\ ABCD = acfhbdeg\}$ *the substitution* $ade\,.\,bfc\,.\,gih$.

Order 9.24. One case is disposed of under the preceding order.

There is a type of primitive group of order 216 *of* 9 *letters. It is composed of the even substitutions of the preceding group.*

In the remaining case the H_{24} of 8 letters is

$$(abcd\,.\,efgh)_{12}(ae\,.\,bf\,.\,cg\,.\,dh).$$

The required group contains, as before, a self-conjugate non-cyclical subgroup of order 9. For one of the substitutions of this we must take

$$\sigma = abc\,.\,egf\,.\,\begin{cases} dhi \\ dih \end{cases}.$$

This is transformed by $abd\,.\,efh$ into

$$\tau = bdc\,.\,fgh\,.\,\begin{cases} aei \\ aie \end{cases}.$$

But

$$\sigma\tau = \begin{cases} adficeh \\ adehcif \end{cases}.$$

Consequently there is no second group of order 9.24.

Order 9.16. The two groups of order 16 of 8 letters available for the present purpose are combinations of the H_8

$$\begin{array}{llll} 1, & ac\,.\,ef\,.\,gh, & ac\,.\,bd\,.\,eg\,.\,fh, & abcd\,.\,efgh, \\ & bd\,.\,eh\,.\,fg, & & adbc\,.\,ehgf, \\ & ab\,.\,cd\,.\,fh, & & \\ & ad\,.\,cb\,.\,eg, & & \end{array}$$

with one of the two substitutions

$$aebfcgdh, \quad afbgchde.$$

In both cases the group of order 9.16 contains a self-conjugate subgroup of order 9.8 composed of the even substitutions. Those of the latter which leave i unchanged are respectively

1, *ac . bd . eg . fh,*	*abcd . efgh,*	1, *ac . bd . eg . fh,*	*abcd . efgh,*
ag . bf . ce . dh,	*adcb . ehgf;*		*adcb . ehgf,*
ae . bh . df . cg,			*ahcf . bgde,*
af . be . ch . dg,			*afch . bedg,*
ah . bg . cf . de,			*agec . bhdf,*
			aecg . bfdh.

The group of order 9 . 8 is readily shown to contain a single, non-cyclical subgroup of order 9. The substitutions of this may be written

$$\begin{array}{llll} 1, & i12 \,.\, 345 \,.\, 678, & i21 \,.\, 354 \,.\, 687, \\ & i36 \,.\, 147 \,.\, 258, & i63 \,.\, 174 \,.\, 285, \\ & i48 \,.\, 237 \,.\, 156, & i84 \,.\, 273 \,.\, 165, \\ & i75 \,.\, 264 \,.\, 183, & i57 \,.\, 246 \,.\, 138. \end{array}$$

On examination it will be found that the only substitution, in 8 letters, of order 2, and not affecting i, which transforms this group into itself is

$$12 \,.\, 36, \quad 48 \,.\, 57.$$

Consequently only the second group of order 16 above is available for the present purpose.

We may then assume $a = 1$, $c = 2$, $b = 3$, $d = 6$. It is then readily found that there is only one type of group of order 9 . 8, viz.:

$$\begin{array}{llll} 1, & iac \,.\, bgh \,.\, dfe, & ica \,.\, bhg \,.\, def, \\ & ibd \,.\, agf \,.\, che, & idb \,.\, afg \,.\, ceh, \\ & ige \,.\, cbf \,.\, ahd, & ieg \,.\, cfb \,.\, adh, \\ & ifh \,.\, cdg \,.\, aeb, & ihf \,.\, cgd \,.\, abe, \end{array}$$

combined with the second H_8 above.

This group is transformed into itself by $ac \,.\, ef \,.\, gh$.

Hence there is a single type of transitive group of order 144 *in* 9 *letters. This is generated by the substitutions*

$$afbgchde, \quad ac \,.\, ef \,.\, gh, \quad iac \,.\, bgh \,.\, dfe.$$

Order 9 . 8. It seems hardly necessary to continue further the detailed explanation of the process by which the possible groups are obtained. I give therefore merely the results in the remaining cases.

There are three types of primitive groups of order 72 *in* 9 *letters. These have for their bases the groups* 8_{20}, 8_{21}, *and* 8_{24} *of* 8 *letters. These are respectively generated by*

(i) *abcd . efgh, ac . fg . eh, iac . bgh . dfe;*

(ii) *afbgchde, iac . bgh . dfe;*

(iii) *abcd . efgh, abcf . bgde, iac . bgh . dfe.*

The last group is composed of the even substitutions of the group of order 9 . 16.

The first group is noteworthy as an instance of a simply transitive primitive group with substitutions leaving more than one letter unchanged. Such a group does not occur for less than 9 letters, and for 9 letters only in this one case.

Order 36. *There is a single type of this order, generated by*

$$abcd\,.\,efgh, \quad iac\,.\,bgh\,.\,dfe.$$

This group is also simply transitive.

The remaining orders do not occur for primitive groups.

The primitive groups of 9 letters are therefore the following. The exponents attached to the order indicate the degree of transitivity.

Order

$9!^{(9)}$	$(abcdefghi)$ all,
$\frac{1}{2}9!^{(7)}$	$(abcdefghi)$ pos.,
$1512^{(3)}$	$\{A(abcdefgh),\ (bcedghf)_{21},\ (aichgbdef)\text{cyc.}\}$,
$504^{(3)}$	$\{A(abcdefgh),\ (bcedghf)\text{cyc.},\ (aichgbdef)\text{cyc.}\}$,
$432^{(2)}$	$\{ade\,.\,bcf,\ acfhbdeg,\ ade\,.\,bfc\,.\,gih\}$,
$216^{(2)}$	$\{ade\,.\,bcf,\ afbe\,.\,chdg,\ ade\,.\,bfc\,.\,gih\}$,
$144^{(2)}$	$\{afbgchde,\ ac\,.\,ef\,.\,gh,\ iac\,.\,bgh\,.\,dfe\}$,
$72^{(1)}$	$\{abcd\,.\,efgh,\ ac\,.\,ef\,.\,gh,\ iac\,.\,bgh\,.\,dfe\}$,
$72^{(2)}$	$\{afbgchde,\ iac\,.\,bgh\,.\,dfe\}$,
$72^{(2)}$	$\{abcd\,.\,efgh,\ ahcf\,.\,bgde,\ iac\,.\,bgh\,.\,dfe\}$,
$36^{(1)}$	$\{abcd\,.\,efgh,\ iac\,.\,bgh\,.\,dfe\}$.

There are accordingly in all $23 + 11 = 34$ transitive groups of 9 letters.

CLASSIFICATION OF MATHEMATICS.

Index du répertoire bibliographique des sciences mathématiques, publié par la Commission permanente du répertoire. Paris, Gauthier-Villars, 1893. 8vo. 14 and 80 pp.

This is a new edition, with but slight modifications, of the proceedings (*procès-verbal sommaire*) of the International Congress for Mathematical Bibliography held at the time of the Paris Exhibition of 1889 (see Bulletin of the New York Mathematical Society, vol. 2, p. 190).

The main part of the pamphlet (pp. 1–80) is occupied with the very elaborate classification of the mathematical sciences which is to serve as "index" to a complete bibliography of modern mathematics (1800–1889). In comparison with the

first edition of 1889, the cross-references are increased in number, some new subdivisions are introduced, and the typographical work is much improved.

This classification is prefaced by a very brief account of the work of the congress of 1889. The resolutions adopted by this congress are given in full; they explain the scope and general plan of the bibliography, which is to be by subjects in logical order and not by authors. Nothing, however, is said about the execution of the plan; nor is there any information given in this new edition as to the progress that may have been made with the work.

The permanent committee charged with the compilation of the *répertoire* is at present (February 15, 1893) constituted as follows: President, Poincaré; Secretary, d'Ocagne; Honorary Members, Prince R. Bonaparte, Prince B. Boncompagni, Darboux, Haton de la Goupillière; Members—from Austria, Em. Weyr; Belgium, Catalan, Le Paige; Denmark, Gram; France, the President of the French Mathematical Society, D. André, Fouret, Charles Henry, G. Humbert, Kœnigs, Laisant, Raffy; Germany, Lampe, Valentin; Great Britain, Glaisher, J. S. Mackay; Greece, Stephanos; Holland, Bierens de Haan, Schoute; Italy, Guccia; Norway, Holst; Portugal, G. Teixeira; Russia, Ligin; Sweden, Eneström; United States, Craig.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, June 3, at half-past three o'clock. In the absence of the president, Professor J. M. Van Vleck occupied the chair. Mr. Charles P. Steinmetz read a paper entitled "On the flow of an incompressible liquid between coaxial cylindrical surfaces." Professor W. Woolsey Johnson communicated a paper entitled "Negative reciprocal equations" by Commander J. E. Craig, U. S. N.

Professor Van Vleck read an extract of a recent letter from Professor Felix Klein, in which Professor Klein stated that he expected to visit America during the present summer and to attend the mathematical congress at Chicago. T. S. F.

THE international congress on mathematics, astronomy, and astro-physics at Chicago will be held in the new Art Institute building on the lake front during the week beginning August 21, 1893. There seems to be every indication of a most successful session. The programme in mathematics will include a series of reviews of the recent development of particular branches of the science. Every one interested in

mathematics will appreciate the importance of attending the congress, although papers will be received from authors even when they cannot be present in person. Full abstracts of papers should be sent in advance to the secretary of the local committee, Professor G. E. Hale, Kenwood Observatory, Chicago.

PROFESSOR FELIX KLEIN of Göttingen at the request of the German government will attend the congress as the official representative of German mathematics. He will deliver several addresses, among them one on "The development of the theory of groups during the last twenty years." Professor Klein has many enthusiastic pupils, friends, and admirers in America (as everywhere), who will rejoice at the opportunity thus afforded of meeting him. He will remain in Chicago during the month of September, and will hold regular conferences with his mathematical friends and any others interested in the recent development of mathematics who may wish to attend. E. H. M.

THE course of lectures upon higher geometry delivered by Professor Klein during the winter semester of 1892–93 has been reproduced in autographic style. It makes a volume of 556 pages, and may be obtained for Mk. 7.50 (including postage), from Dr. Fr. Schilling, assistant at the royal collection of mathematical models, Göttingen.

WE are indebted to Dr. J. C. Fields for a copy of *L'Indicateur des Cours publics* of Paris for the second semester 1892–93. Among the mathematical courses announced at the *Sorbonne* are the following: Appell, Dynamics of systems; Boussinesq, Theory of the waves of oscillation; Hermite, Theory of Eulerian integrals and elliptic functions; Lippmann, Electricity; Picard, Surfaces of Riemann and abelian integrals; Poincaré, Thermodynamics; Wolf, Progress of astronomy. At the *College de France* the mathematical courses are: Deprez, Applications of thermodynamics; Jordan, Complex numbers; Koenigs, Geodesic lines; Mascart, Electricity.

THE *Grand prix des sciences mathématiques* of 1892 for the determination of the number of primes inferior to a given limit was awarded to J. Hadamard. The *Prix Bordin* offered for the application of the general theory of abelian functions to geometry was awarded to G. Humbert. The *Prix Bordin* offered for the question first proposed in 1888, to study the surfaces whose linear element can be reduced to the form

$$ds^2 = [f(u) - \phi(v)]\,(du^2 + dv^2)\,,$$

was awarded to Gabriel Kœnigs, with honorable mentions to Otto Ohnesorge and Louis Raffy.

For the grand prize of 1894 the subject of competition will be to perfect in an important point the theory of the deformation of surfaces. The *Prix Bordin* of the same year is offered for the study of those problems of mechanics which involve integrals algebraic with respect to the velocities, and, in particular, quadratic integrals.

THE Royal Academy of Belgium has announced the following subjects for its mathematical and physical prizes to be awarded in 1894 : (1) Exposition and discussion of the various theories of diffusion of one liquid into another, with new facts bearing on this; (2) Estimate of theories explaining the constitution of solutions; new experiments throwing light on the subject, and especially on the existence of hydrates in aqueous solutions; (3) The investigations of modern geometers on the theory of the triple orthogonal system to be summarized and extended in some important respect.

THE publication of a new periodical, "*L'Intermédiaire des Mathématiciens*," will be begun by Messrs. Gauthier-Villars in January, 1894. The editors will be Dr. C. A. Laisant and Mr. Émile Lemoine. The object of the journal will be to furnish information to its readers upon any mathematical question or subject in which they may be interested. A circular has been issued in which the editors ask instructors, students, and others interested in mathematics, to use it as a means of seeking any information desired, and to cooperate in the labor of answering the questions proposed. It will appear monthly. The subscription price, including postage, is six francs. T. S. F.

DR. GEORGE W. HILL has accepted an invitation from Columbia College to give a course of lectures on celestial mechanics under the auspices of the department of astronomy. The lectures will begin October 14, 1893, and will continue weekly throughout the college year. Persons desiring to attend these lectures can secure information by addressing the professor of astronomy at Columbia College. J. K. R.

THE forty-second meeting of the American Association for the Advancement of Science, which is to be held this year at Madison, Wisconsin, will take place August 17 to 24.

THE following courses in mathematics are offered to graduate students at Johns Hopkins University during the academic year 1893–94 :—Professor Craig: Theory of functions

(elementary course), twice weekly through the year; Theory of functions (advanced course), twice weekly through the year; Differential equations, twice weekly through the year; Algebraic integrals of one and two variables, three times weekly, first half-year; Elliptic functions, three times weekly, second half-year; Mathematical seminary, weekly through the year. Professor Franklin: Theory of Algebraic forms, three times weekly, first half-year; Exercises in analytical geometry of two and three dimensions, twice weekly, first half-year; Theory of numbers, three times weekly, second half-year; Theory of probability, twice weekly, second half-year. Mr. Hulburt: Theory of substitutions with applications to algebraic equations, three times weekly, first half-year; General theory of plane algebraic curves, three times weekly, second half-year.

In astronomy the courses will be as follows:—Professor Newcomb: The method of least squares, twice weekly, first half-year; Astronomical optics, twice weekly, first half-year; Advanced theoretical astronomy, twice weekly, second half-year. Dr. Poor: General course in theoretical and practical astronomy, three times weekly, through the year; Methods of computing orbits, ephemerides, and special perturbations, three times weekly through the year; Astronomical seminary, weekly through the year; Practical work with the instruments, daily through the year.

We have to record the death of Professor Heinrich Durège at Prague on April 19 in his seventy-second year.

On May 14, at Berlin, died Ernst Eduard Kummer, professor of mathematics in the University of Berlin, aged 84 years. T. S. F.

The University of Chicago offers the following advanced mathematical courses during 1893–94, each being four hours a week for twelve weeks, or, if enclosed in brackets, for twenty-four weeks: Theta-functions [Functions of a complex variable] —Professor Moore. Hyperelliptic functions, Theory of substitutions [Advanced integral calculus]—Professor Bolza. Finite groups of linear substitutions, Line geometry, Geometry of surfaces, Analytical mechanics, Theory of potential, Mathematical electricity—Professor Naschke. [Determinants and advanced theory of equations]—Dr. Younge. Differential equations—Dr. Boyd.

The mathematical club and seminary hold bi-weekly meetings for the review of books and memoirs and for the presentation of the results of research. E. H. M.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). A short history of mathematics. 2d ed. London, Macmillan, 1893. 8vo. 530 pp. 10*s*.

BOHL (P.). Ueber die Darstellung von Functionen einer Variabeln durch trigonometrische Reihen mit mehreren einer Variabeln proportionalen Argumenten. Dorpat, 1893. 4to. 31 pp. Mk. 1.20

CHINI (M.). Esercizi di calcolo infinitesimale. Livorno, 1893. 12mo. 270 pp. Mk. 3.50.

DESCARTES. See Goldbreck.

EUCLIDES.—Codex Leidensis 399, 1. Euclidis elementa ex interpretatione Al-Hadschdschadii cum commentariis Al-Narizii. Arabice et latine ediderunt notisque instruxerunt *R. O. Besthorn* et *J. L. Heiberg*. Volumen I. Fasciculus 1. Havniæ, 1893. 8vo. pp. 1–92. Mk. 5.00

FORSYTH (A. R.). A treatise on the theory of functions of a complex variable. London, Clay (Cambridge Warehouse), 1893. (New York, Macmillan.) Royal 8vo. 22 and 682 pp. (21*s*. net.) $8.50

GEIGENMÜLLER (R.). Elemente der höheren Mathematik. 2te Aufl. Band II.: Die niedere und höhere Analysis mit Rücksicht auf Functionen einer reellen Urvariablen. Mittweida, 1893. 8vo. 16 and 336 pp. 2 plates. Mk. 7.00

GOLDBRECK (E.). Descartes' mathematisches Wissenschaftsideal. Berlin, 1893. 8vo. 42 pp. Mk. 1.20

GÜNTSCHE (R.). Beitrag zur Integrirung einer Differentialgleichung. Berlin, 1893. 4to. Mk. 1.00

HAAG (P.). Cours de calcul différentiel et intégral. Paris, 1893. 8vo. 5 and 619 pp. Ill.

HERCHER (B.). Lehrbuch der analytischen Geometrie der Ebene. Leipzig, 1893. 8vo. Ill. Mk. 0.75

HUYGHENS (C.). Œuvres complètes, publiées par la Société Hollandaise des Sciences. Tome V: Correspondance 1664–65. La Haye, 1893. 4to. 625 pp. 3 plates. Ill. Mk. 25.00

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KOCH (H. v.). Sur les déterminants infinis et les équations différentielles linéaires. Upsal, 1892. 4to. 79 pp. Mk. 2.50

LANGE (E.). Zeichnung des neunten Schnittpunktes zweier Curven dritter Ordnung. Wismar, 1893. 4to. 20 pp. 4 plates. Mk. 1.80

MERRIMAN (M.). Metodo de los cuadrados minimos. Libro de testo. Traducido del Ingles para V. Balbin. Buenos Aires, 1889. 8vo. 12 and 208 pp.

MUNK (M.). Ueber die möglichen Fälle mehrfach hyperboloidischer Lagen zweier Tetraiden. Marburg, 1893. 8vo. 51 pp. Mk. 1.50

OTT (A.). Ueber Determinanten. Weimar, 1893. 4to. 26 pp. Mk. 1.50

RITTER (E.). Die eindeutigen automorphen Formen vom Geschlechte Null. Eine Revision und Erweiterung der Poincaré'schen Sätze. Göttingen, 1892. 8vo. 82 pp. Ill. Mk. 1.80

SCHWENDENWEIN (H.). Das regelmässige 257-Eck. Teschen, 1892. 8vo. 22 pp. Mk. 1.20

SUHLE (H.). Ueber imaginäre Punkte ebener Curven. Dessau, 1893. 4to. 28 pp. Mk. 1.20

II. ELEMENTARY MATHEMATICS.

AUGUST (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 18te Auflage, besorgt durch F. August. Leipzig, 1893. 8vo. 8 and 204 pp. Cl. Mk. 1.60

BENSEMANN (H.). Die constructive Methode im planimetrischen Unterricht. Cöthen, 1893. 4to. 25 pp. Mk. 1.20

BRADBURY (W. F.). Academic geometry, plane and solid. Boston, 1893. 8vo. 374 pp. with figures. Half leather. $1.50

DANZIG (E.). Uebungsstoff zur Auflösung planimetrischer Constructionsaufgaben mittelst algebraischer Analysis. Leipzig, 1893. 8vo. 79 pp. Mk. 1.20

EASTON (J. G.). First book of algebra for young beginners. With numerous easy examples and answers. London, Newman, 1893. 12mo. 180 pp. 2*s*. 6*d*.

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 39te Aufl. Halle, 1893. 8vo. 166 and 53 pp. Cl. Mk. 2.50

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HANUS (P. H.). Geometry in the grammar school: an essay; with illustrative class exercises, and an outline of the work for the last three years of the grammar school. Boston, Heath, 1893. 12mo. 4 and 52 pp. $0.25

HELLWIG (C.). Berechnung der Wurzeln cubischer und biquadratischer Gleichungen, besonders auch der irrationalen Wurzeln der ersteren im irreducibien Falle. Leipzig, 1893. 8vo. 28 pp. Mk. 1.00

KOPPE (M.). Die Behandlung der Logarithmen im Unterricht. Berlin, 1893. 4to. Mk. 1.00

LANGE (J.). Synthetische Geometrie der Kegelschnitte, nebst Uebungsaufgaben. Berlin, H. W. Miller, 1893. 8vo. 68 pp. Ill. Mk. 1.20

MEYER (A.). Die Grundlehren der ebenen Trigonometrie nach den neuen preussischen Lehrplänen. Cöthen, 1893. 4to. 21 pp. 2 plates. Mk. 1.50

PANIZZA (F.). Aritmetica razionale. 2a ed. riveduta. Milano, Hoepli, 1893. 222 pp. Fr. 1.50

SAUVAGE (P.). Les lieux géométriques en géométrie élémentaire. Paris, 1893. 8vo. Ill. Mk. 2.80

SCHÜLKE (A.). Trigonometrische Tafel. Leipzig, 1893. Mk. 0.10

SMITH (C.). A treatise on algebra. 4th ed. London, Macmillan, 1893. 8vo. 660 pp. 7*s.* 6*d.*

TAYLOR (J. M.). An academic algebra. Boston, Allyn & Bacon, 1893. $1.00

III. APPLIED MATHEMATICS.

BAZALA (J.). Beleuchtungsconstructionen für windschiefe Schraubenflächen. Bielitz, 1892. 8vo. 16 pp. 1 plate. Mk. 1.20

BEZIAT D'AUDIBERT (E.). Théorie élémentaire des assurances sur la vie et autres opérations viagères. 2d ed. Paris, 1893. 8vo 204 pp. Fr. 10.00

BLASCHKE (E.). Die Methoden der Ausgleichung von Massenerscheinungen mit besonderer Berücksichtigung der Ausgleichung von Absterbe- und Invalidenordnungen. Wien, 1893. 8vo. 7 and 122 pp. 4 plates. Mk. 5.40

BLAIKIE (J.). Elements of dynamics (mechanics). With numerous examples and examination questions. New and enlarged ed. 8vo. Edinburgh, Thin, 1893. 196 pp. 3*s.* 6*d.*

BORLETTI (F.). Celerimensura e tavole logaritmiche à 4 decimali. Milano, Hoepli, 1893. 16mo. 6 and 145 pp. Ill. Fr. 3.50

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CRESCINI (E.). Sul moto di un sistema rigido che si appoggia su di una superficie fissa. Pisa, Pieraccini, 1892. 8vo. 30 pp. [Supplemento al *Nuovo Cimento*, serie III, vol. XXXII, fasc. di settembre e ottobre 1892.]

FABRI (CORNELIA). Sul moti vorticosi nei fluidi perfetti. Bologna, Gamberini, 1892. 4to. 67 pp.

FOUSSEREAU (G.). Polarisation rotatoire, réflexion et réfraction vitreuses, réflexion métallique. Leçons faites à la Sorbonne en 1891-2. Rédigées par J. Lemoine. Paris, 1893. 8vo. 350 pp. Ill. Mk. 10.00

FRITSCHE (H.). Ueber die Bestimmung der geographischen Länge und Breite und der drei Elemente des Erdmagnetismus durch Beobachtung zu Lande, sowie erdmagnetische und geographische Messungen an mehr als tausend verschiedenen Orten in Asien und Europa 1867-91. St. Petersburg, 1893. 8vo. 109 pp. 4 plates and maps. Mk. 3.00

GELDARD (C.). Statics and dynamics. New York, Longmans, 1893. 8vo. 12 and 308 pp. Ill. Cl. $1.50

HERZOG (J.) und FELDMANN (C. F.). Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. Berlin, 1893. 8vo. 8 and 364 pp. Ill. Cl. Mk. 12.00

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JAERISCH (P.). Zur Theorie der elastischen Kugelwellen, mit Anwendung auf die Reflexion und Brechung des Lichtes. Hamburg, 1893. 4to. 45 pp. Mk. 1.80

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——— A Case of Non-Euclidean Geometry. *Bulletin of the New York Mathematical Society*, vol. 2, pp. 158–161, April, 1893.

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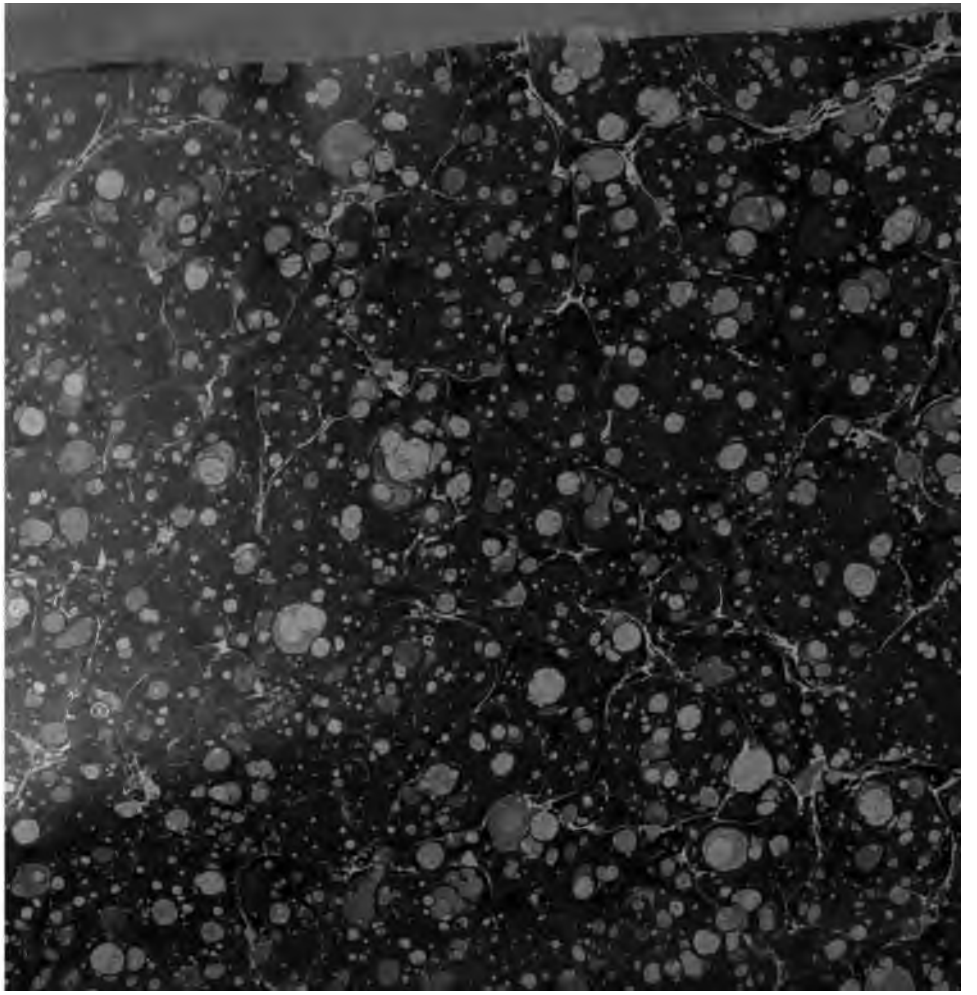

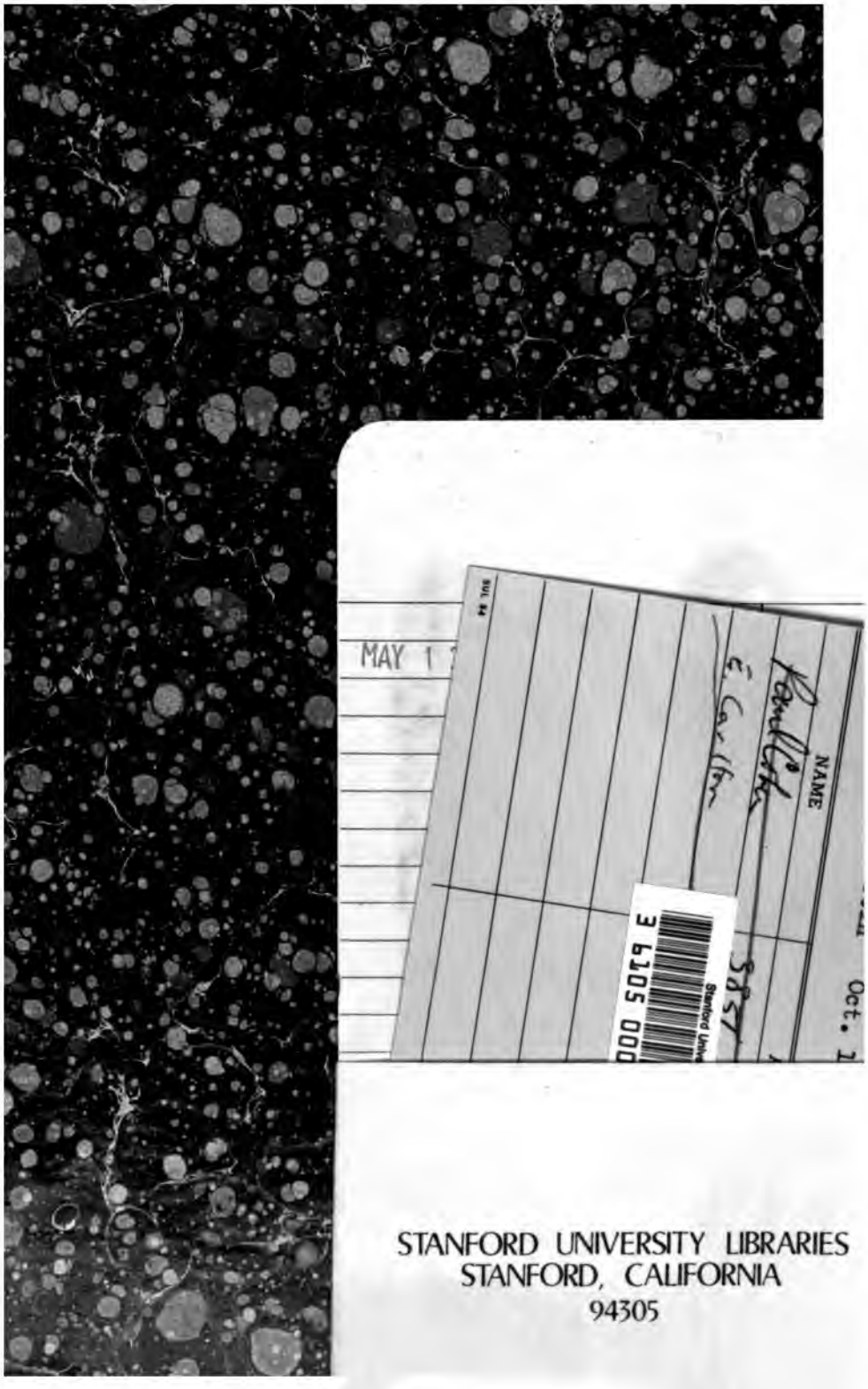

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